Problema 8: Să se calculeze următoarele limite:

a) $\lim\limits_{x\to\infty}\left(x+x^2\right)$; b) $\lim\limits_{x\to\infty}\left(-x-x^2\right)$;

c) $\lim\limits_{x\to-\infty}\left(x-x^2\right)$; d) $\lim\limits_{x\to-\infty}\left(-x-x^3\right)$;

e) $\lim\limits_{x\to\infty}\left(x+2\right)$; f) $\lim\limits_{x\to-\infty}\left(-x+2\right)$;

g) $\lim\limits_{x\to\infty}x^x$; h) $\lim\limits_{x\to\infty}x^{1-x}$;

Soluţie parţială:

g) $\lim\limits_{x\to\infty}x^x \overset{\infty^\infty}{=} \infty$; h) $\lim\limits_{x\to\infty}x^{1-x} \overset{\infty^{-\infty}}{=} 0$;

Problema 9: Să se calculeze următoarele limite:

a) $\lim\limits_{x\to1}(1-x^3+x)$; b) $\lim\limits_{x\to-\infty}(1-x^3+x)$;

c) $\lim\limits_{x\to5}7$ d) $\lim\limits_{x\to7}0$

e) $\lim\limits_{x\to5}\left(2x+1\right)$ f) $\lim\limits_{x\to2}\left(x^2-1\right)$

g) $\lim\limits_{x\to\infty}\left(x^2+2x-1\right)$ h) $\lim\limits_{x\to\infty}\left(-x^2+2x-1\right)$

i) $\lim\limits_{x\to\infty}\left(x^3+2x-1\right)$ j) $\lim\limits_{x\to\infty}\left(-x^3+2x-1\right)$

Soluţie parţială:

a) $\lim\limits_{x\to1}\left(1-x^3+x\right)=1-1+1=1$; b) $\lim\limits_{x\to-\infty}\left(1-x^3+x\right)=\lim\limits_{x\to-\infty}\left(-x^3\right)=\infty$;

Problema 10: Studiaţi existenţa limitele următoare, şi în caz afirmativ, calculaţi limitele respective:

a) $\lim\limits_{x\to0}\dfrac{1}{x^2}$; b) $\lim\limits_{x\to1}\dfrac{-1}{(x-1)^2}$;

c) $\lim\limits_{x\to0}\dfrac{1}{x^3}$; d) $\lim\limits_{x\to1}\dfrac{1}{1-x^2}$;

e) $\lim\limits_{x\to\infty}\dfrac{1}{x^2}$; f) $\lim\limits_{x\to-\infty}\dfrac{1}{x^3}$;

g) $\lim\limits_{x\to1}\dfrac{x+1}{1-2x^3}$; h) $\lim\limits_{x\to-\infty}\dfrac{x+1}{1-2x^3}$;

i) $\lim\limits_{x\to 1}\dfrac{x^3+1}{1-2x^3}$;   j) $\lim\limits_{x\to -\infty}\dfrac{x^3+1}{1-2x^3}$;

k) $\lim\limits_{x\to 1}\dfrac{x^3+1}{1-2x}$;   l) $\lim\limits_{x\to -\infty}\dfrac{x^3+1}{1-2x}$;

m) $\lim\limits_{x\to 5}\dfrac{3x+1}{x-4}$ n) $\lim\limits_{x\to 5}\dfrac{x^2-4x-1}{x-4}$

o) $\lim\limits_{x\to \infty}\dfrac{3x+1}{x-4}$ p) $\lim\limits_{x\to -\infty}\dfrac{3x+1}{x-4}$

q) $\lim\limits_{x\to \infty}\dfrac{3x+1}{x^2-4}$ r) $\lim\limits_{x\to -\infty}\dfrac{3x+1}{x^2-4}$

s) $\lim\limits_{x\to \infty}\dfrac{3x^3+1}{x^2-4}$ ş) $\lim\limits_{x\to -\infty}\dfrac{3x^3+1}{x^2-4}$

t) $\lim\limits_{x\to 2}\dfrac{3x^3+1}{x^2-4}$ ţ) $\lim\limits_{x\to 2}\dfrac{3x^3+1}{4-x^2}$

u) $\lim\limits_{x\to 2}\dfrac{3x^3+1}{(x-2)^2}$ v) $\lim\limits_{x\to 2}\dfrac{3x^3+1}{(x-2)^3}$

Soluţie parţială:

a) $\lim\limits_{x\to 0}\dfrac{1}{x^2}\overset{\frac{1}{+0}}{=}\infty$; b) $\lim\limits_{x\to 1}\dfrac{-1}{(x-1)^2}\overset{\frac{-1}{+0}}{=}-\infty$; c) $\lim\limits_{x\to 0}\dfrac{1}{x^3}\overset{\frac{1}{?0}}{=}$, însă trebuie să găsim semnul lui 0. Facem tabelul de semne $\Rightarrow$

x	0
$\dfrac{1}{x^3}$	+ + + + + \ - - - - - - -

$\Rightarrow \lim\limits_{\substack{x\to 0\\ x<0}}\dfrac{1}{x^3}\overset{\frac{1}{-0}}{=}-\infty$ şi $\lim\limits_{\substack{x\to 0\\ x>0}}\dfrac{1}{x^3}\overset{\frac{1}{+0}}{=}\infty$,

aşadar $\dfrac{1}{x^3}$ nu are limită în x = 0.

d) $\lim\limits_{x\to 1}\dfrac{1}{1-x^2}\overset{\frac{1}{?0}}{=}$

x		-1		1	
1	+ + + + + + +	+	+ + + + + + + + + +	+	+ + + +
$1-x^2$	- - - - - -	0	+ + + + + + + + + +	0	- - - -
$\dfrac{1}{1-x^2}$	- - - - - - -	/	+ + + + + + + + + +	/	- - - -

$\lim\limits_{\substack{x\to 1\\ x<1}} \dfrac{1}{1-x^2} \overset{\frac{1}{+0}}{=} \infty$; $\lim\limits_{\substack{x\to 1\\ x>1}} \dfrac{1}{1-x^2} \overset{\frac{1}{-0}}{=} -\infty \Rightarrow \nexists \lim\limits_{x>1} \dfrac{1}{1-x^2}$; e) $\lim\limits_{x\to\infty} \dfrac{1}{x^2} \overset{\frac{1}{\infty}}{=} 0$;

c) $\lim\limits_{x\to-\infty} \dfrac{1}{x^3} \overset{\frac{1}{-\infty}}{=} 0$;

g) $\lim\limits_{x\to 1} \dfrac{x+1}{1-2x^3} = \dfrac{1+1}{1-2} = -2$;

h) $\lim\limits_{x\to-\infty} \dfrac{x+1}{1-2x^3} = 0$;

i) $\lim\limits_{x\to 1} \dfrac{x^3+1}{1-2x^3} = \dfrac{1+1}{1-2} = -2$;

j) $\lim\limits_{x\to-\infty} \dfrac{x^3+1}{1-2x^3} = \dfrac{1}{-2} = -\dfrac{1}{2}$;

k) $\lim\limits_{x\to 1} \dfrac{x^3+1}{1-2x} = \dfrac{1+1}{1-2} = -2$;

Problema 11: Studiaţi existenţa limitele următoare, şi în caz afirmativ, calculaţi limitele respective:

a) $\lim\limits_{x\to 1} 2^x$;

b) $\lim\limits_{x\to 0} 2^x$;

c) $\lim\limits_{x\to\infty} (-2)^x$;

d) $\lim\limits_{x\to\infty} (-1)^x$

e) $\lim\limits_{x\to\infty} \left(-\dfrac{2}{3}\right)^x$;

f) $\lim\limits_{x\to\infty} \left(\dfrac{2}{3}\right)^x$

g) $\lim\limits_{x\to\infty} 1^x$;

h) $\lim\limits_{x\to\infty} 3^x$;

i) $\lim\limits_{x\to-\infty} (-2)^x$;

j) $\lim\limits_{x\to-\infty} (-1)^x$

k) $\lim_{x\to-\infty}\left(-\frac{2}{3}\right)^x$;

l) $\lim_{x\to-\infty}\left(\frac{2}{3}\right)^x$

m) $\lim_{x\to-\infty}1^x$;

n) $\lim_{x\to-\infty}3^x$;

Problema 12: Studiaţi existenţa limitele următoare, şi în caz afirmativ, calculaţi limitele respective:

a) $\lim_{x\to1}\log_2 x$;

b) $\lim_{x\to2}\log_2 x$;

c) $\lim_{\substack{x\to0\\x>0}}\log_2 x$;

d) $\lim_{\substack{x\to0\\x>0}}\log_{\frac{1}{2}} x$

e) $\lim_{x\to\infty}\log_2 x$;

f) $\lim_{x\to\infty}\log_{\frac{1}{2}} x$

g) $\lim_{x\to1}\log_3 x$;

h) $\lim_{x\to3}\log_3 x$;

i) $\lim_{\substack{x\to0\\x>0}}\log_3 x$;

j) $\lim_{\substack{x\to0\\x>0}}\log_{\frac{1}{3}} x$

k) $\lim_{x\to\infty}\log_3 x$;

l) $\lim_{x\to\infty}\log_{\frac{1}{3}} x$

Soluţie parţială:

a) $\lim_{x\to1}\log_2 x=0$;

c) $\lim_{\substack{x\to0\\x>0}}\log_2 x=-\infty$;

d) $\lim_{\substack{x\to0\\x>0}}\log_{\frac{1}{2}} x=\infty$;

e) $\lim_{x\to\infty}\log_2 x=\infty$;

f) $\lim_{x\to\infty}\log_{\frac{1}{2}} x=-\infty$;

Problema 13: Studiaţi existenţa limitele următoare, şi în caz afirmativ, calculaţi limitele respective:

a) $\lim_{x\to0}\sin x$

b) $\lim_{x\to0}\cos x$

c) $\lim_{x\to\frac{\pi}{2}}\sin x$

d) $\lim_{x\to\frac{\pi}{2}}\cos x$

e) $\lim_{x\to\infty}\sin x$

f) $\lim_{x\to\infty}\cos x$

g) $\lim_{x\to-\infty}\sin x$

h) $\lim_{x\to-\infty}\cos x$

i) $\lim\limits_{\substack{x\to\pi/2\\ x<\pi/2}} tgx$

j) $\lim\limits_{\substack{x\to\pi/2\\ x>\pi/2}} tgx$

k) $\lim\limits_{\substack{x\to 0\\ x>0}} ctgx$

l) $\lim\limits_{\substack{x\to \pi\\ x<\pi}} ctgx$

m) $\lim\limits_{x\to\infty} arctgx$

n) $\lim\limits_{x\to-\infty} arctgx$

o) $\lim\limits_{x\to\infty} arcctgx$

o) $\lim\limits_{x\to-\infty} arcctgx$

Problema 14: Studiaţi existenţa limitele următoare, şi în caz afirmativ, calculaţi limitele respective:

a) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$

b) $\lim\limits_{x\to 0}\dfrac{x}{\sin x}$

c) $\lim\limits_{x\to 1}\dfrac{\sin(\ln x)}{\ln x}$

d) $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 5x}$

e) $\lim\limits_{x\to 1}\dfrac{x^2-1}{\sin \pi x}$

f) $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\cos x}{2x-\pi}$

g) $\lim\limits_{x\to 0}\dfrac{2x}{tgx}$

h) $\lim\limits_{x\to 0}\dfrac{tg3x}{\sin 5x}$

i) $\lim\limits_{x\to 0}\dfrac{arctg3x}{\sin 5x}$

j) $\lim\limits_{x\to 0}\dfrac{arctg3x}{tg5x}$

k) $\lim\limits_{x\to 2}\dfrac{arctg(x-2)}{x^2-4}$

l) $\lim\limits_{x\to 0}\dfrac{\sin^3 x}{\arcsin(x^5)}$

Problema 15: Studiaţi existenţa limitele următoare, şi în caz afirmativ, calculaţi limitele respective:

a) $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}$

b) $\lim\limits_{x\to\infty}\dfrac{2\ln x}{x^2}$

c) $\lim\limits_{x\to\infty}\dfrac{\ln(x^3+x+1)}{x^3}$

d) $\lim\limits_{x\to\infty}\dfrac{\ln(x^3+x+1)}{x}$

e) $\lim\limits_{x\to\infty}\dfrac{\ln(1+x^2)}{x}$

f) $\lim\limits_{x\to\infty}\dfrac{\ln\left(\sqrt{x^5}\right)}{x}$

g) $\lim\limits_{\substack{x\to\frac{\pi}{2} \\ x<\frac{\pi}{2}}} \frac{\ln tgx}{tgx}$ h) $\lim\limits_{\substack{x\to 0 \\ x>0}} \frac{\ln ctgx}{ctgx}$

Problema 16: Studiaţi existenţa limitele următoare, şi în caz afirmativ, calculaţi limitele respective:

a) $\lim\limits_{x\to\infty} \frac{e^x}{x}$ b) $\lim\limits_{x\to\infty} \frac{e^{\ln x}}{\ln x}$

c) $\lim\limits_{x\to\infty} \frac{e^{x^3+x+1}}{x^3+x^2}$ d) $\lim\limits_{x\to\infty} \frac{e^{x^3+x+1}}{2x^3+x^2}$

e) $\lim\limits_{x\to\infty} \frac{e^{x^3+x+1}}{x^2+x}$ f) $\lim\limits_{x\to\infty} \frac{e^{x^3+x+1}}{-x^2+x}$

g) $\lim\limits_{x\to\infty} \frac{e^{x^3+x+1}}{-2x^2+x}$ h) $\lim\limits_{x\to\infty} \frac{e^{\ln\left(x^3+x+1\right)}}{\ln\left(\frac{1}{x^3+x+1}\right)}$

Problema 17: Studiaţi existenţa limitele următoare, şi în caz afirmativ, calculaţi limitele respective:

a) $\lim\limits_{x\to 0} \frac{2^x-1}{x}$ b) $\lim\limits_{x\to 0} \frac{2^{-x}-1}{x}$

c) $\lim\limits_{x\to 0} \frac{2^{-x}-1}{3x}$ d) $\lim\limits_{x\to 0} \frac{2^x-2}{3x}$

e) $\lim\limits_{x\to 0} \frac{2^{\sin x}-1}{\sin x}$ f) $\lim\limits_{x\to 0} \frac{2^{\sin x}-1}{x}$

g) $\lim\limits_{x\to 0} \frac{2^x-1}{3^x-1}$ h) $\lim\limits_{x\to 0} \frac{2^{\sin x}-1}{3^x-1}$

Problema 18: Studiaţi existenţa limitele următoare, şi în caz afirmativ, calculaţi limitele respective:

a) $\lim\limits_{x\to 0} \frac{\ln\left(1+x\right)}{x}$ b) $\lim\limits_{x\to 0} \frac{\ln\left(1+x^2\right)}{x^2}$

c) $\lim\limits_{x\to1}\dfrac{\ln x}{x-1}$

d) $\lim\limits_{x\to1}\dfrac{\ln x}{x^2-1}$

e) $\lim\limits_{x\to0}\dfrac{\ln(1+\sin x)}{\sin x}$

f) $\lim\limits_{x\to0}\dfrac{\ln(1+\sin x)}{x}$

g) $\lim\limits_{x\to1}\dfrac{\ln(1+\ln x)}{\ln x}$

h) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\ln(1+\ln(\sin x))}{\ln(\sin x)}$

Problema 19: Studiaţi existenţa limitele următoare, şi în caz afirmativ, calculaţi limitele respective:

a) $\lim\limits_{x\to0}\dfrac{(1+x)^3-1}{x}$

b) $\lim\limits_{x\to1}\dfrac{(1+\ln x)^5-1}{\ln x}$

c) $\lim\limits_{x\to1}\dfrac{(1+\ln x)^7-1}{x}$

d) $\lim\limits_{x\to0}\dfrac{\left(1+\sqrt{x}\right)^4-1}{x}$

e) $\lim\limits_{x\to-\infty}\dfrac{\left(1+3^x\right)^2-1}{3^x}$

f) $\lim\limits_{x\to\infty}\dfrac{\left(1+3^{-x}\right)^2-1}{3^{-x}}$

g) $\lim\limits_{x\to-\infty}\dfrac{\left(1+3^x\right)^2-1}{2^x}$

h) $\lim\limits_{x\to\infty}\dfrac{5\left(1+3^{-x}\right)^2-5}{7^{-x}}$

Problema 20: Studiaţi existenţa limitele următoare, şi în caz afirmativ, calculaţi limitele respective:

a) $\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}$;

b) $\lim\limits_{x\to0}(1+3x)^{\frac{1}{x}}$;

c) $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x}\right)^{2x}$;

d) $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+1}\right)^{3x}$;

e) $\lim\limits_{x\to\infty}\left(\dfrac{\sqrt{x}+1}{\sqrt{x}}\right)^{x}$;

f) $\lim\limits_{x\to0}(1+\sin x)^{\frac{1}{x}}$;

g) $\lim\limits_{x\to0}\left(\dfrac{2^x+3^x}{2}\right)^{\frac{1}{x}}$;

h) $\lim\limits_{x\to0}\left(\dfrac{2^x+3^x+4^x}{3}\right)^{\frac{1}{x}}$;

Problema 21: Calculaţi limitele următoare în cazul $\infty-\infty$:

a) $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$ b) $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2+3}\right)$

c) $\lim_{x\to\infty}\left(3^x-5^x\right)$ d) $\lim_{x\to\infty}\left(3^x-x\right)$

e) $\lim_{x\to\infty}\left(\ln x-x\right)$ f) $\lim_{x\to\infty}\left(\ln\left(x+1\right)-\ln x\right)$

g) $\lim_{x\to\infty}\left(\sqrt[3]{x^3+1}-\sqrt{x^2+1}\right)$ h) $\lim_{x\to\infty}\left(\sqrt{x+1}-2\sqrt{x}+\sqrt{x+2}\right)$

Problema 22: Calculaţi limitele următoare în cazul $\frac{\infty}{\infty}$:

a) $\lim_{x\to\infty}\frac{5x^3+x+2}{x^2+x-1}$ b) $\lim_{x\to\infty}\frac{x+2}{\sqrt{x^2+x-1}}$

c) $\lim_{x\to\infty}\frac{x^2+\ln x}{x^2-\ln x}$ d) $\lim_{x\to\infty}\frac{\ln\left(1+3^x\right)}{\ln\left(1+4^x\right)}$

e) $\lim_{x\to\infty}\frac{3^x}{4^x+1}$ f) $\lim_{x\to\infty}\frac{3^x-5^x}{4^x+2^x}$

g) $\lim_{x\to-\infty}\frac{x+2}{\sqrt{x^2+x-1}}$ h) $\lim_{x\to-\infty}\frac{\sqrt{4x^2+2}}{\sqrt[3]{x^3+x-1}}$

Problema 23: Calculaţi limitele următoare în cazul $\frac{0}{0}$:

a) $\lim_{x\to1}\frac{x^3-1}{x^2-1}$ b) $\lim_{x\to1}\frac{x^4-1}{x-1}$

c) $\lim_{x\to0}\frac{\sin 2x}{tg5x}$ d) $\lim_{x\to0}\frac{\sin\left(tg5x\right)}{\sin\left(\sin 3x\right)}$

e) $\lim_{x\to0}\frac{\ln\left(1+2x\right)}{tg5x}$ f) $\lim_{x\to0}\frac{4^x-2^x}{x}$

g) $\lim_{x\to\pi}\frac{\sin 3x}{\sin 4x}$ h) $\lim_{x\to\frac{\pi}{2}}\frac{tg3x}{\sin 4x}$

i) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$ j) $\lim\limits_{x\to 1}\dfrac{\sqrt{x+8}-3}{\sqrt{x+3}-2}$

Problema 24: Calculaţi limitele următoare în cazul $0\cdot\infty,\ \infty\cdot 0$:

a) $\lim\limits_{x\to\infty} x\cdot\ln\left(1+\dfrac{1}{x}\right)$ b) $\lim\limits_{x\to\infty}(x+1)\cdot\left(3^{\frac{1}{x+2}}-1\right)$

c) $\lim\limits_{x\to\infty} x\cdot\sin\dfrac{1}{x}$ d) $\lim\limits_{x\to 0} x\cdot ctg\dfrac{1}{x}$

e) $\lim\limits_{x\to\infty} x\cdot\left[\ln(x+1)-\ln(x-1)\right]$ f) $\lim\limits_{x\to\infty} x\cdot\left[\sqrt{x+1}-\sqrt{x}\right]$

g) $\lim\limits_{x\to\infty} x\cdot\left(2^{\frac{1}{x}}-1\right)$ h) $\lim\limits_{x\to\infty} x^2\cdot\left[e^{\frac{1}{x}}-e^{\frac{1}{x+1}}\right]$

Problema 25: Calculaţi limitele următoare în cazul 1^{∞}:

a) $\lim\limits_{x\to 0}(1+3x)^{\frac{2}{x}}$; b) $\lim\limits_{x\to 0}(1-3x)^{\frac{2}{x}}$;

c) $\lim\limits_{\substack{x\to 0\\ x>0}}\left(\cos\sqrt{x}\right)^{\frac{1}{x}}$; d) $\lim\limits_{x\to 0}(2-\cos x)^{\frac{3}{4x^2}}$;

e) $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$; f) $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$;

g) $\lim\limits_{x\to 0}\left(\dfrac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}}$. h) $\lim\limits_{x\to 3}(7-2x)^{tg\frac{\pi x}{6}}$;

Problema 26: Calculaţi limitele următoare în cazul $0^{\infty};0^{0};\infty^{0}$:

a) $\lim\limits_{\substack{x\to 0\\ x>0}} x^{x}$; b) $\lim\limits_{\substack{x\to 0\\ x>0}}(\sin x)^{x}$

c) $\lim\limits_{\substack{x\to 0\\ x>0}} x^{\sin x}$; d) $\lim\limits_{x\to\infty} x^{\frac{1}{\sqrt{x}}}$

e) $\lim_{x\to\infty}\left(\frac{1}{x}\right)^{x}$;

f) $\lim_{x\to\infty} x^{\frac{1}{x}}$;

g) $\lim_{\substack{x\to 0\\ x>0}} x^{\frac{1}{x}}$;

h) $\lim_{\substack{x\to 0\\ x>0}}\left(\frac{1}{x}\right)^{x}$;

Problema 27. Calculaţi următoarele limite:

a) $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}$;

b) $\lim_{x\to \pi}\frac{\sin 2x}{\sin 3x}$;

c) $\lim_{x\to 1}\frac{tg\left(2(x-1)\right)}{tg\left(3(x-1)\right)}$;

d) $\lim_{x\to 2}\frac{\arcsin(x-2)}{x^2-4}$;

e) $\lim_{x\to 3}\frac{x^3-27}{arctg\,(x-3)}$;

f) $\lim_{x\to 1}\frac{2^{x-1}-1}{x^2-1}$;

g) $\lim_{x\to 1}\frac{\ln x}{1-x^2}$;

h) $\lim_{x\to 0} x\cdot\sin\frac{1}{x}$;

Soluţie parţială:

a) $\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{3x}{\sin 3x}\cdot\frac{2}{3}=1\cdot 1\cdot\frac{2}{3}=\frac{2}{3}$;

b) $\lim_{x\to \pi}\frac{\sin 2x}{\sin 3x}\underset{\overline{\overline{}}}{\overset{t=x-\pi}{}}\lim_{t\to 0}\frac{\sin 2(t+\pi)}{\sin 3(t+\pi)}=$

$=\lim_{t\to 0}\frac{\sin 2t\cdot\cos 2\pi+\cos 2t\cdot\sin 2\pi}{\sin 3t\cdot\cos 3\pi+\cos 3t\cdot\sin 3\pi}=\lim_{t\to 0}\frac{\sin 2t}{-\sin 3t}=$

$=\lim_{t\to 0}\frac{\sin 2t}{2t}\cdot\frac{3t}{\sin 3t}\cdot\frac{2}{-3}=-\frac{2}{3}$;

c) $\lim_{x\to 1}\frac{tg\left(2(x-1)\right)}{tg\left(3(x-1)\right)}\overset{x-1=t}{=\!=}\lim_{t\to 0}\frac{tg2t}{tg3t}=\lim_{t\to 0}\frac{tg2t}{2t}\cdot\frac{3t}{tg3t}\cdot\frac{2}{3}=1$;

d) $\lim_{x\to 2}\frac{\arcsin(x-2)}{x-2}\cdot\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}$;

e) $\lim\limits_{x\to 3}\dfrac{(x-3)(x^2+3x+9)}{arctg\,(x-3)}=\lim\limits_{x\to 3}(x^2+3x+9)=27$;

f) $\lim\limits_{x\to 1}\dfrac{2^{x-1}-1}{x-1}\cdot\dfrac{x-1}{(x-1)(x+1)}=\lim\limits_{x\to 1}\ln 2\cdot\dfrac{1}{x+1}=\dfrac{1}{2}\ln 2$;

g) $\lim\limits_{x\to 1}\dfrac{\ln x}{1-x^2}\underline{\underline{t=x-1}}\lim\limits_{t\to 0}\dfrac{\ln(t+1)}{1-(1+t)^2}=\lim\limits_{t\to 0}\dfrac{\ln(1+t)}{-2t-t^2}=$

$=\lim\limits_{t\to o}\dfrac{\ln(1+t)}{t}\cdot\dfrac{1}{-2-t}=-\dfrac{1}{2}$;

h) $\lim\limits_{x\to 0}x\sin\dfrac{1}{x}=0$ căci $x\to 0$ iar $\sin\dfrac{1}{x}$ mărginit.

Problema 28. Calculaţi următoarele limite:

a) $\lim\limits_{x\to 7}\dfrac{2-\sqrt{x-3}}{x^2-49}$;

b) $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}+\dfrac{3}{1-x^3}\right)$;

c) $\lim\limits_{x\to -\infty}x\left(\sqrt{4x^2+7}+2x\right)$;

d) $\lim\limits_{x\to -\infty}x^3\left(e^{\frac{1}{x}}-e^{\frac{1}{x+1}}\right)$;

e) $\lim\limits_{x\to 0}\left(1+\ln(1+x)+\ln(1+2x)+....+\ln(1+nx)\right)^{\frac{1}{x}}$, x>0;

f) $\lim\limits_{x\to \infty}x\left(\ln(1+x)-\ln x\right)$;

g) $\lim\limits_{\substack{x\to 0\\ x>0}}\dfrac{x\cdot e^{-\frac{1}{x}}}{tg^2x}$;

h) $\lim\limits_{x\to 0}\dfrac{e^{x^3}-1}{\sin^3 x}$;

i) $\lim\limits_{x\to 0}\dfrac{x^2\cdot\cos\frac{1}{x}}{\sin x}$;

j) $\lim\limits_{x\to 3}(7-2x)^{tg\frac{\pi x}{6}}$;

k) $\lim\limits_{x\to \infty}\left(\cos\dfrac{2\pi x}{x+1}\right)^{x^2}$;

l) $\lim\limits_{x\to 0}\dfrac{\ln(1+x+x^2)+\ln(1-x+x^2)}{x^2}$;

m) $\lim\limits_{\substack{x\to 0\\ x>0}}\left(\frac{1}{x^2}-ctg^2 x\right)$; n) $\lim\limits_{x\to 0}\frac{x^n-(\sin x)^n}{x^{n+2}}$;

Soluţie parţială:

a) $l=\lim\limits_{x\to 7}\frac{\left(2-\sqrt{x-3}\right)\left(2+\sqrt{x-3}\right)}{(x-7)(x+7)\left(2+\sqrt{x-3}\right)}=\lim\limits_{x\to 7}\frac{4-x+3}{(x-7)(x+7)\left(2+\sqrt{x-3}\right)}=$

$=\lim\limits_{x\to 7}\frac{-1}{(x+7)\left(2+\sqrt{x-3}\right)}=\frac{-1}{14(2+2)}=-\frac{1}{56}$;

b) $\lim\limits_{x\to 1}\left(\frac{1}{x-1}-\frac{3}{(x-1)\left(1+x+x^2\right)}\right)=\lim\limits_{x\to 1}\frac{1+x+x^2-3}{(x-1)\left(x^2+x+1\right)}=$

$\lim\limits_{x\to 1}\frac{x^2+x-2}{(x-1)\left(x^2+x+1\right)}=\lim\limits_{x\to 1}\frac{(x-1)(x+2)}{(x-1)\left(x^2+x+1\right)}=\lim\limits_{x\to 1}\frac{x+2}{x^2+x+1}=\frac{3}{3}=1$

c) $\lim\limits_{x\to -\infty}x\left(\sqrt{4x^2+7}+2x\right)$. Notând $-x=t\Rightarrow$

$l=\lim\limits_{t\to\infty}-t\left(\sqrt{4t^2+7}-2t\right)=\lim\limits_{t\to\infty}-t\cdot\frac{4t^2+7-4t^2}{\sqrt{4t^2+7}+2t}=$

$=\lim\limits_{t\to\infty}-\frac{7t}{t\left(\sqrt{4+\frac{7}{t^2}}+2\right)}=-\frac{7}{4}$;

d) $\lim\limits_{x\to -\infty}x^3\left(e^{\frac{1}{x}}-e^{\frac{1}{x+1}}\right)\overset{t=-x}{=\!=\!=}\lim\limits_{t\to\infty}(-t)^3\left(e^{\frac{-1}{t}}-e^{\frac{1}{1-t}}\right)=$

$=\lim\limits_{t\to\infty}-t^3\cdot e^{\frac{1}{1-t}}\cdot\left(e^{\frac{1}{-t}-\frac{1}{1-t}}-1\right)=\lim\limits_{t\to\infty}-t^3\cdot e^{\frac{1}{1-t}}\cdot\frac{e^{\frac{1}{t(t-1)}}-1}{\frac{1}{t(t-1)}}\cdot\frac{1}{t(t-1)}=$

$=\lim\limits_{t\to\infty}-\frac{t^2}{t-1}\cdot e^{\frac{1}{1-t}}\cdot 1=-\infty$;

e) $l=\lim\limits_{x\to 0}\left(1+\ln((1+x)(1+2x)...(1+nx)\right)^{\frac{1}{x}}=$

$=\lim\limits_{x\to o}\left[\left(1+\ln(f(x)\right)^{\frac{1}{f(x)}}\right]^{\frac{f(x)}{x}}$, unde

$f(x)=\ln(1+x)(1+2x)\ldots(1+nx)\Rightarrow l=e^{\lim\limits_{x\to 0}\frac{f(x)}{x}}$.

Fie $l_1=\lim\limits_{x\to 0}\dfrac{f(x)}{x}=\lim\limits_{x\to 0}\left(\dfrac{\ln(1+x)}{x}+\dfrac{\ln(1+2x)}{x}+...+\dfrac{\ln(1+nx)}{x}\right)=$

$=\lim\limits_{x\to 0}\left(\dfrac{\ln(1+x)}{x}+\dfrac{\ln(1+2x)}{2x}\cdot 2+...+\dfrac{\ln(1+nx)}{nx}\cdot n\right)=$

$=1+2+...+n=\dfrac{n(n+1)}{2}\Rightarrow l=e^{\frac{n(n+1)}{2}}$.

f) $\lim\limits_{x\to\infty}x\cdot\ln\dfrac{1+x}{x}=\lim\limits_{x\to\infty}\ln\left(1+\dfrac{1}{x}\right)^{x}=\ln e=1$;

g) $\lim\limits_{\substack{x\to 0\\ x>0}}\dfrac{xe^{-\frac{1}{x}}}{tg^2 x}=\lim\limits_{\substack{x\to 0\\ x>0}}\left(\dfrac{x}{tgx}\right)^2\cdot\dfrac{e^{\frac{-1}{x}}}{x}$. Notăm $\dfrac{1}{x}=t\Rightarrow l=\lim\limits_{t\to\infty}\dfrac{e^{-t}}{\frac{1}{t}}=$

$=\lim\limits_{t\to\infty}\dfrac{t}{e^t}=0$;

h) $\lim\limits_{x\to 0}\dfrac{e^{x^3}-1}{\sin^3 x}=\lim\limits_{x\to 0}\dfrac{e^{x^3}-1}{x^3}\cdot\left(\dfrac{x}{\sin x}\right)^3=1$;

i) $\lim\limits_{x\to 0}\left(\dfrac{x}{\sin x}\right)\cdot x\cdot\cos\dfrac{1}{x}=\lim\limits_{x\to 0}x\cdot\cos\dfrac{1}{x}=0$ căci $x\to 0$ iar $\cos\dfrac{1}{x}$ e mărginit .

j) $\lim\limits_{x\to 3}(7-2x)^{tg\frac{x\pi}{6}}$. Notând $t = x-3 \Rightarrow$

$$l=\lim_{t\to 0}\left(7-2(t+3)\right)^{tg\left(\frac{\pi t}{6}+\frac{\pi}{2}\right)}=\lim_{t\to 0}(1-2t)^{tg\left(\frac{\pi}{2}-\left(-\frac{\pi t}{6}\right)\right)}=$$

$$=\lim_{t\to 0}\left[(1+(-2t))^{\frac{1}{-2t}}\right]^{-2t\cdot ctg\left(-\frac{\pi t}{6}\right)}=e^{\lim\limits_{t\to 0}-2t\cdot\frac{1}{tg\left(-\frac{\pi t}{6}\right)}}=e^{l_1}.$$

$$l_1=\lim_{t\to 0}\frac{-2t}{-tg\,\frac{\pi t}{6}}=\lim_{t\to 0}\frac{\frac{\pi t}{6}}{tg\,\frac{\pi t}{6}}\cdot\frac{6}{\pi t}\cdot 2t=\frac{12}{\pi}\Rightarrow l=e^{\frac{12}{\pi}}.$$

k) $\lim\limits_{x\to\infty}\left(\cos\dfrac{2\pi x}{x+1}\right)^{x^2}$. Deoarece $\dfrac{2\pi x}{x+1}\to 2\pi$ iar $\cos 2\pi = 1,$ suntem în cazul 1^{∞}. Folosind $\cos(2\pi-t)=\cos t \Rightarrow \cos\dfrac{2\pi x}{x+1}=\cos\left(2\pi-\dfrac{2\pi x}{x+1}\right)=$

$$=\cos\frac{2\pi x+2\pi-2\pi x}{x+1}=\cos\frac{2\pi}{x+1}$$

$$l=\lim_{x\to\infty}\left(1+\cos\frac{2\pi}{x+1}-1\right)^{x^2}=\lim_{x\to\infty}\left(1-2\sin^2\frac{\pi}{x+1}\right)^{x^2}=$$

$$=\lim_{x\to\infty}\left[\left(1+\left(-2\sin^2\frac{\pi}{x+1}\right)\right)^{\frac{1}{-2\sin^2\frac{\pi}{x+1}}}\right]^{(-2\sin^2\frac{\pi}{x+1})\cdot x^2}=e^{l_1}.$$ Calculăm

$$l_1=\lim_{x\to\infty}-2\left(\frac{\sin\frac{\pi}{x+1}}{\frac{\pi}{x+1}}\right)^2\cdot\left(\frac{\pi}{x+1}\right)^2\cdot x^2=-2\pi^2\Rightarrow l=e^{-2\pi^2}\quad .$$

Obs: S-a folosit $\cos 2t = 1-2\sin^2 t$.

l) $\lim\limits_{x\to 0}\dfrac{\ln\left(1+x+x^2\right)+\ln\left(1-x+x^2\right)}{x^2}=$

$=\lim\limits_{x\to 0}\dfrac{\ln\left(\left(1+x+x^2\right)\left(1-x+x^2\right)\right)}{x^2}=\lim\limits_{x\to 0}\dfrac{\ln\left(\left(1+x^2\right)^2-x^2\right)}{x^2}=$

$=\lim\limits_{x\to 0}\dfrac{\ln\left(1+x^2+x^4\right)}{x^2+x^4}\cdot\dfrac{x^2+x^4}{x^2}=\lim\limits_{x\to 0}1\left(1+x^2\right)=1$

Obs: Trebuie evitate greşelile de tipul :

$l=\lim\limits_{x\to 0}\left(\dfrac{\ln\left(1+x+x^2\right)}{x+x^2}\cdot\dfrac{x+x^2}{x^2}+\dfrac{\ln\left(1-x+x^2\right)}{-x+x^2}\cdot\dfrac{-x+x^2}{x^2}\right)=$

$=\lim\limits_{x\to 0}\left(\dfrac{1+x}{x}+\dfrac{-1+x}{x}\right)=\lim\limits_{x\to 0}\dfrac{1+x-1+x}{x}=2$, fals deoarece am trecut „parţial” la limită .

Problema 29: Determinaţi constantele $a,b\in\mathbb{R}$ ce apar în limitele următoare pentru a obţine rezultatele cerute :

a) $\lim\limits_{x\to 1}\dfrac{\sqrt{x^2+3x+a}-b}{x^2+x-2}=\dfrac{5}{18}$;

b) $\lim\limits_{x\to -\infty}\left(\sqrt[3]{8x^3-ax^2}-bx+2\right)=1$;

c) $\lim\limits_{x\to\infty}\left(a+\dfrac{bx}{x^2-1}\right)^x=e^2$;

d) $\lim\limits_{x\to 0}\dfrac{e^{ax^2}-\cos x}{x^2}=\dfrac{3}{2}$.

Soluţie parţială:

a) $l=\dfrac{\sqrt{4+a}-b}{0}$, iar pentru ca $l=\dfrac{5}{18}$ este necesar ca

$\sqrt{4+a}=b \Leftrightarrow b\geq 0$ si $a+4=b^2$

$$\Rightarrow l=\lim_{x\to 1}\frac{\sqrt{x^2+3x+b^2-4}-b}{(x-1)(x+2)}=$$

$$=\lim_{x\to 1}\frac{x^2+3x+b^2-4-b^2}{(x-1)(x+2)\left(\sqrt{x^2+3x+b^2-4}+b\right)}=$$

$$=\lim_{x\to 1}\frac{x^2+3x-4}{(x-1)(x+2)(\sqrt{x^2+3x+b^2-4}+b)}=$$

$$=\lim_{x\to 1}\frac{(x-1)(x+4)}{(x-1)(x+2)(\sqrt{x^2+3x+b^2-4}+b)}=$$

$$=\lim_{x\to 1}\frac{x+4}{(x+2)(\sqrt{x^2+3x+b^2-4}+b)}=\frac{5}{3(\sqrt{b^2}+b)}=\frac{5}{6b}.$$

Dar, $l=\dfrac{5}{18}\Leftrightarrow 6b=18\Leftrightarrow b=3$ si $a=5$.

Se verifică apoi că pentru a=5 si b=3 limita este cea cerută.

b) $\lim\limits_{x\to -\infty}\left(\sqrt[3]{8x^3-ax^2}-bx\right)=1-2$. Notăm $t=-x \Leftrightarrow$

$$\Leftrightarrow \lim_{t\to\infty}\left(bt+\sqrt[3]{-8t^3-at^2}\right)=-1 \Leftrightarrow \lim_{t\to\infty}\left(bt-\sqrt[3]{8t^3+at^2}\right)=-1.$$

Dar, $l=\lim\limits_{t\to\infty} t\left(b-\sqrt[3]{8+\dfrac{a}{t}}\right)=\infty(b-2)$. Pentru ca $l\in\mathbb{R}$ trebuie ca b=2 , altfel $l=\pm\infty$. Deci b = 2 $\Rightarrow \lim\limits_{t\to\infty}\left(2t-\sqrt[3]{8t^3+at^2}\right)=-1 \Leftrightarrow$

$$\lim_{t\to\infty}\frac{8t^3-8t^3-at^2}{4t^2+2t\sqrt[3]{8t^3+at^2}+\sqrt[3]{(8t^3+at^2)^2}}=-1$$

$$\Leftrightarrow \lim_{t\to\infty}\frac{-at^2}{t^2\left(4+2\sqrt[3]{8+\dfrac{a}{t}}+\sqrt[3]{\left(8+\dfrac{a}{t}\right)^2}\right)}=-1 \Leftrightarrow \frac{-a}{4+4+4}=-1 \Leftrightarrow$$

$a=12$. Asadar , conditii necesare sunt a=12 ; b=2 . Se verifica apoi ca $\lim\limits_{x\to-\infty}\left(\sqrt[3]{8x^3-12x^2}-2x\right)=-1\Rightarrow$ a =12 ; b=2 sunt suficiente.

c) $\lim\limits_{x\to\infty}\left(a+\dfrac{bx}{x^2-1}\right)^x=e^2$. Deoarece $\dfrac{bx}{x^2-1}\to 0,\forall b\in\mathbb{R}$, trebuie ca $a^x\to e^2$ .Pentru $a\le -1\Rightarrow$ nu $\exists$ limita, pentru $a\in(-1;1)\Rightarrow l=0$, pentru $a>1\Rightarrow l=\infty$ .Singurul caz posibil este a=1 , deci am obtinut o conditie necesara. $l=\lim\limits_{x\to\infty}\left[\left(1+\dfrac{bx}{x^2-1}\right)^{\frac{x^2-1}{bx}}\right]^{\frac{bx}{x^2-1}\cdot x}=e^{l_1}$ unde

$l_1=\lim\limits_{x\to\infty}\dfrac{bx^2}{x^2-1}=b\quad\Rightarrow l=e^b=e^2\Leftrightarrow b=2$. Se verifica a=1,b=2 sunt conditii suficiente.

d) $l=\lim\limits_{x\to0}\left(\dfrac{e^{ax^2}-1}{ax^2}\cdot a+\dfrac{1-\cos x}{x^2}\right)=a+\lim\limits_{x\to0}\dfrac{2\sin^2\frac{x}{2}}{x^2}=$

$$=a+\lim_{x\to0}2\cdot\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\cdot\frac{x^2}{4}\cdot\frac{1}{x^2}=a+\frac{1}{2}=\frac{3}{2}\Rightarrow a=1$$

Problema 30: Calculaţi

a) $\lim\limits_{x\to0}\dfrac{e^{x^2}-\cos x}{x^2}$

b) $\lim\limits_{x\to0}\dfrac{1-\cos x\cos 2x....\cos nx}{x^2},n\in\mathbb{N}^*$

c) $\lim\limits_{x\to1}\left(\dfrac{m}{1-x^m}-\dfrac{n}{1-x^n}\right),m,n\in\mathbb{N}^*$

d) $\lim\limits_{x\to-\infty}x^3\left(e^{\frac{1}{x}}-e^{\frac{1}{x+1}}\right)$

e) $\lim\limits_{x\to\infty}\left(\dfrac{2^x+3^x+4^x}{3}\right)^{\frac{1}{x}}$

f) $\lim\limits_{x\to0}\dfrac{(1+x)^{\frac{1}{x}}-e}{x}$

g) $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$ h) $\lim_{x\to\infty}\left(x-x^2\ln\frac{1+x}{x}\right)$

Problema 31: Calculaţi

a) $\lim_{x\to 0} x\sin\frac{1}{x}$ b) $\lim_{x\to\infty}\left(\sin\sqrt{x+1}-\sin\sqrt{x}\right)$

c) $\lim_{x\to\pi}\frac{\sin mx}{\sin nx}, m,n\in\mathbb{N}^*$ d) $\lim_{x\to\pi}\frac{\sin x}{1-\frac{x^2}{\pi^2}}$

e) $\lim_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{x+1}}{arctg\frac{1}{x}-arctg\frac{1}{x+1}}$ f) $\lim_{x\to 0}\frac{x^2\cos\frac{1}{x}}{\sin x}$

g) $\lim_{x\to 0}\frac{1-\cos(n\arcsin n)}{x^2}, n\in\mathbb{N}^*$ h) $\lim_{x\to 0}\frac{tg(ax)-\sin(ax)}{tg(bx)-\sin(bx)}, a,b\in\mathbb{R}^*$

Problema 32: Calculaţi

a) $\lim_{x\to 0}\frac{e^{x^3}-1}{\sin^3 x}$ b) $\lim_{x\to\infty}\left(\frac{1+x}{2+x}\right)^{\frac{1-x}{1-\sqrt{x}}}$

c) $\lim_{x\to 3}\left(7-2x\right)^{tg\frac{\pi x}{6}}$ d) $\lim_{x\to\infty}\left(\cos\frac{2\pi x}{x+1}\right)^{x^2}$

e) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{x}{x-\sin x}}$ f) $\lim_{x\to\infty}\left(\frac{\pi}{2}-arctgx\right)^{\frac{1}{\ln x}}$

g) $\lim_{\substack{x\to 0\\ x>0}}\left(\frac{1}{x^2}-ctg^2x\right)$ h) $\lim_{x\to 0}\frac{x^n-(\sin x)^n}{x^{n+2}}, n\in\mathbb{N}^*$