PROBLEME NONSTANDARD CU MATRICI

1) Fie $A=\begin{pmatrix}-1 & 2\\ -3 & 4\end{pmatrix}$ și $B=\begin{pmatrix}5 & -4\\ 3 & -2\end{pmatrix}$

demonstrați că, $\forall n\geq 1$ avem

$A^n-B^n=(2^n-1)(A-B)$

2) Fie $A=\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}$, se cere

A^n, $n\geq 1$

3) se cere $A^n=?$, $n\geq 2$ dacă

$A=\begin{pmatrix}2 & 0 & 1\\ 0 & -3 & 0\\ 1 & 0 & 2\end{pmatrix}$

4) se cere $A^n=?$, $n\geq 2$ dacă

$A=\begin{pmatrix}1+\sqrt{3} & 1-\sqrt{3}\\ \sqrt{3}-1 & \sqrt{3}+1\end{pmatrix}$

5) Fie $A=\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{pmatrix}$,

se cere A^n

Calculați A^n, $n \geq 2$ pt $A=\begin{pmatrix}2&0&1\\0&-3&0\\1&0&2\end{pmatrix}$

$E_1)$ $A^2=A\cdot A=\begin{pmatrix}2&0&1\\0&-3&0\\1&0&2\end{pmatrix}\begin{pmatrix}2&0&1\\0&-3&0\\1&0&2\end{pmatrix}=\begin{pmatrix}5&0&4\\0&9&0\\4&0&5\end{pmatrix}$ și nu avem regulă

$E_2)$ $A=I_3+B$ unde $B=A-I_3 \Rightarrow B=\begin{pmatrix}1&0&1\\0&-4&0\\1&0&1\end{pmatrix}$

$E_3)$ $B^2=B\cdot B=\begin{pmatrix}1&0&1\\0&-4&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&-4&0\\1&0&1\end{pmatrix}=\begin{pmatrix}2&0&2\\0&16&0\\2&0&2\end{pmatrix}$, $B^3=\begin{pmatrix}4&0&4\\0&-64&0\\4&0&4\end{pmatrix}$

și deducem că $B^n=\begin{pmatrix}2^{n-1}&0&2^{n-1}\\0&(-4)^n&0\\2^{n-1}&0&2^{n-1}\end{pmatrix}$, $\forall n\geq 1$ și se arată prin inducție că e adevărată

$E_4)$ $A^n=(I_3+B)^n=C_n^0 I_3^n B^0+C_n^1 I_3^{n-1}B+C_n^2 I_3^{n-2}B^2+\dots+C_n^n I_3^{n-n}B^n$

$=C_n^0 I_3+C_n^1 B+C_n^2 B^2+\dots+C_n^n B^n=$

$=C_n^0\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+C_n^1\begin{pmatrix}1&0&1\\0&-4&0\\1&0&1\end{pmatrix}+C_n^2\begin{pmatrix}2&0&2\\0&16&0\\2&0&2\end{pmatrix}+\dots+C_n^n\begin{pmatrix}2^{n-1}&0&2^{n-1}\\0&(-4)^n&0\\2^{n-1}&0&2^{n-1}\end{pmatrix}$

$\Rightarrow A^n=\begin{pmatrix}C_n^0+C_n^1+C_n^2\cdot 2+\dots+C_n^n 2^{n-1}&0&0+C_n^1+C_n^2\cdot 2+\dots+C_n^n 2^{n-1}\\0&C_n^0+C_n^1(-4)+\dots+C_n^n(-4)^n&0\\0+C_n^1+C_n^2\cdot 2+\dots+C_n^n 2^{n-1}&0&C_n^0+C_n^1+C_n^2\cdot 2+\dots+C_n^n 2^{n-1}\end{pmatrix}$

$E_5)$ Fie $S_1=C_n^1+C_n^2\cdot 2+\dots+C_n^n 2^{n-1}\,/\cdot 2 \Rightarrow 2S_1=C_n^1 2+C_n^2 2^2+\dots+C_n^n 2^n\,/+C_n^0$

$\Rightarrow C_n^0+2S_1=C_n^0+C_n^1 2+C_n^2 2^2+\dots+C_n^n 2^n \Rightarrow 1+2S_1=(1+2)^n \Rightarrow$

$1+2S_1=3^n \Rightarrow S_1=\frac{3^n-1}{2} \Rightarrow$ term $a_{11}=1+\frac{3^n-1}{2}=\boxed{\frac{3^n+1}{2}}$

$E_6)$ Fie $S_2=C_n^0+C_n^1\cdot(-4)+C_n^2(-4)^2+\dots+C_n^n(-4)^n \Rightarrow S_2=(1-4)^n=(-3)^n \Rightarrow a_{22}=(-3)^n$

$E_7)$ a_{13} este $S_1=\frac{3^n-1}{2} \Rightarrow$

$$A^n=\begin{pmatrix}\frac{3^n+1}{2}&0&\frac{3^n-1}{2}\\0&(-3)^n&0\\\frac{3^n-1}{2}&0&\frac{3^n+1}{2}\end{pmatrix}$$

Calculați A^n, $n \geq 2$ pt $A=\begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ \sqrt{3}-1 & \sqrt{3}+1 \end{pmatrix}$

$E_1)$ $A=\begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ -(1-\sqrt{3}) & 1+\sqrt{3} \end{pmatrix} \Rightarrow A=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, încercăm

să o scriem trigonometric sub forma $A=\alpha\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$

unde $\alpha=\sqrt{a^2+b^2} \Rightarrow \alpha=\sqrt{(1+\sqrt{3})^2+(1-\sqrt{3})^2}=\sqrt{1+2\sqrt{3}+3+1-2\sqrt{3}+3}$

$\Rightarrow \alpha=\sqrt{8}=2\sqrt{2} \Rightarrow A=2\sqrt{2}\begin{pmatrix} \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{2}} \\ -\frac{1-\sqrt{3}}{2\sqrt{2}} & \frac{1+\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$

$\Rightarrow A=2\sqrt{2}\begin{pmatrix} \frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \\ -\frac{\sqrt{2}-\sqrt{6}}{4} & \frac{\sqrt{2}+\sqrt{6}}{4} \end{pmatrix}$

$E_2)$ ar trebui $\cos\varphi=\frac{\sqrt{2}+\sqrt{2}\cdot\sqrt{3}}{4} \Rightarrow \cos\varphi=\frac{1}{2}\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}$,

dar $\cos\varphi=\cos x\cos y+\sin x\sin y \Rightarrow \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}$

$\Rightarrow \varphi=\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12} \Rightarrow A=2\sqrt{2}\begin{pmatrix} \cos\frac{\pi}{12} & \sin\frac{\pi}{12} \\ -\sin\frac{\pi}{12} & \cos\frac{\pi}{12} \end{pmatrix}$

$E_3)$ A are forma $A=\alpha\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$, se demonstrează

prin inducție (dar aici nu mai am loc) că $A^n=\alpha^n\begin{pmatrix} \cos n\varphi & \sin n\varphi \\ -\sin n\varphi & \cos n\varphi \end{pmatrix}$,

$\Rightarrow A^n=(2\sqrt{2})^n\begin{pmatrix} \cos\frac{n\cdot\pi}{12} & \sin\frac{n\pi}{12} \\ -\sin\frac{n\pi}{12} & \cos\frac{n\pi}{12} \end{pmatrix}$

Fie $A=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$, calculați A^n, $\forall n\geq 1$

E1) încercăm $A^2=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}=\begin{pmatrix}6&5&5\\5&6&5\\5&5&6\end{pmatrix}$ și calculând $A^3 \Rightarrow$ nu găsim regulă

E2) $A=I_3+B$ unde $B=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

E3) $B^2=B\cdot B=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}=\begin{pmatrix}3&3&3\\3&3&3\\3&3&3\end{pmatrix}=3B$

$B^3=B^2\cdot B=3B\cdot B=3B^2=3\cdot 3B=3^2B$, analog $B^n=3^{n-1}B$

E4) $A^n=(I_3+B)^n=C_n^0 I_3^n B^0+C_n^1 I_3^{n-1}B+C_n^2 I_3^{n-2}B^2+\dots+C_n^n I_3^0 B^n$

Folosind $I_3^k=I_3$, $B^0=I_3$ și că $B^k=3^{k-1}B$

E5) $A^n=C_n^0 I_3+C_n^1 B+C_n^2\, 3B+\dots+C_n^n 3^{n-1}B$

$A^n=C_n^0 I_3+B(C_n^1+C_n^2\, 3+\dots+C_n^n 3^{n-1})$

E6) Pt a forma Newton în paranteză $\Rightarrow$

$A^n=C_n^0 I_3+B(C_n^1 3+C_n^2 3^2+\dots+C_n^n 3^n)\cdot\frac{1}{3}$

$A^n=C_n^0 I_3+B(\underbrace{C_n^0+C_n^1 3+C_n^2 3^2+\dots+C_n^n 3^n}_{(1+3)^n}-\underbrace{C_n^0}_{1})\cdot\frac{1}{3}$

$A^n=C_n^0 I_3+B\cdot\frac{4^n-1}{3}\Rightarrow A^n=1\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\frac{4^n-1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

$\Rightarrow A^n=\begin{pmatrix}1+\frac{4^n-1}{3}&\frac{4^n-1}{3}&\frac{4^n-1}{3}\\ - & - & -\end{pmatrix}=\begin{pmatrix}\frac{4^n+2}{3}&\frac{4^n-1}{3}&\frac{4^n-1}{3}\\ \frac{4^n-1}{3}&\frac{4^n+2}{3}&\frac{4^n-1}{3}\\ \frac{4^n-1}{3}&\frac{4^n-1}{3}&\frac{4^n+2}{3}\end{pmatrix}$