CALCULUL INVERSEI UNEI MATRICI

Calculați (în caz că există) inversa următoarelor matrici:

1) $A=\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$

2) $B=\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

3) $C=\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 2 & 2 & 5 \end{pmatrix}$

4) $D=\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

5) $E=\begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 2 & 1 \end{pmatrix}$

6) $F=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

7) $G=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

8) $H=\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

9) $I=\begin{pmatrix} 1 & -1 & 2 \\ -2 & 3 & -1 \\ 4 & -2 & 5 \end{pmatrix}$

10) $J=\begin{pmatrix} 4 & 2 & -4 \\ 3 & 2 & -1 \\ -2 & -1 & 5 \end{pmatrix}$

11) Fie $A=\begin{pmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 9 & 6 & 3 \end{pmatrix}$ și $B=A+I_3$, arătați că B este inversabilă și $B^{-1}=I_3-\frac{1}{11}A$

12) Fie $A_4=\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$, arătați că A_4 are inversă și că inversa sa este

$$B=\begin{pmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

Fie $A=\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}$, calculați $A^{-1}=?$

$E_1)$ $\det A = 9-8 = 1 \neq 0 \Rightarrow \exists A^{-1}$

$E_2)$ $A^t=\begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix}$

$E_3)$ $A^*=\begin{pmatrix}a^*_{11} & a^*_{12}\\ a^*_{21} & a^*_{22}\end{pmatrix}$ unde a^*_{ij} se găsește din A^t

$E_4)$ $a^*_{11}=(-1)^{1+1}\cdot 3 = 3$; $\quad a^*_{12}=(-1)^{1+2}\cdot 4 = -4$

$a^*_{21}=(-1)^{2+1}\cdot 2 = -2$; $\quad a^*_{22}=(-1)^{2+2}\cdot 3 = 3$

$E_5)$ $A^*=\begin{pmatrix}3 & -4\\ -2 & 3\end{pmatrix}$

$E_6)$ $A^{-1}=\dfrac{1}{\det A}\cdot A^* = \dfrac{1}{1}\cdot\begin{pmatrix}3 & -4\\ -2 & 3\end{pmatrix}=\begin{pmatrix}3 & -4\\ -2 & 3\end{pmatrix}$

Fie $B=\begin{pmatrix}1 & 0 & -1\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$, se cere B^{-1}

$E_1)$ $\det B=\begin{vmatrix}1 & 0 & -1\\ 0 & 0 & -1\\ 0 & 1 & 0\end{vmatrix}=0+0+0-0+1-0=1\neq 0$
$\Rightarrow \exists B^{-1}$

$E_2)$ $B^t=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ -1 & -1 & 0\end{pmatrix}$

$E_3)$ $B^*=\begin{pmatrix}1 & -1 & 0\\ 0 & 0 & +1\\ 0 & -1 & 0\end{pmatrix}$

$b^*_{11}=(-1)^{1+1}\cdot\begin{vmatrix}0 & 1\\ -1 & 0\end{vmatrix}=1\cdot(0+1)=1$

$b^*_{12}=(-1)^{1+2}\cdot\begin{vmatrix}0 & 1\\ -1 & 0\end{vmatrix}=-1\cdot(0+1)=-1$

$b^*_{13}=(-1)^{1+3}\cdot\begin{vmatrix}0 & 0\\ -1 & -1\end{vmatrix}=0$; $b^*_{21}=(-1)^{2+1}\cdot\begin{vmatrix}0 & 0\\ -1 & 0\end{vmatrix}=0$

$b^*_{22}=(-1)^{2+2}\cdot\begin{vmatrix}1 & 0\\ -1 & 0\end{vmatrix}=1\cdot 0=0$ $b^*_{23}=(-1)^{2+3}\cdot\begin{vmatrix}1 & 0\\ -1 & -1\end{vmatrix}=$
$=-1\cdot(-1)=1$

$b^*_{31}=(-1)^{3+1}\cdot\begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix}=0$; $b^*_{32}=(-1)^{3+2}\cdot\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=-1$; $b^*_{33}=0$

$E_4)$ $B^{-1}=\dfrac{1}{\det B}\cdot B^*=\dfrac{1}{1}\cdot\begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}=\begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}$

Fie $I = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 3 & -1 \\ 4 & -2 & 5 \end{pmatrix}$, se cere I^{-1}

$E_1)$ $\det I = \begin{vmatrix} 1 & -1 & 2 \\ -2 & 3 & -1 \\ 4 & -2 & 5 \end{vmatrix} = 15+4+8-24-2-10 =$
$= 27 - 36 = -9$

$E_2)$ $I^t = \begin{pmatrix} 1 & -2 & 4 \\ -1 & 3 & -2 \\ 2 & -1 & 5 \end{pmatrix}$

$E_3)$ $i^*_{1,1} = (-1)^{1+1} \cdot \begin{vmatrix} 3 & -2 \\ -1 & 5 \end{vmatrix} = 13$; $i^*_{1,2} = (-1)^{1+2} \cdot \begin{vmatrix} -1 & -2 \\ 2 & 5 \end{vmatrix} = 1$

$i^*_{1,3} = (-1)^{1+3} \cdot \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} = -5$; $i^*_{2,1} = (-1)^{2+1} \cdot \begin{vmatrix} -2 & 4 \\ -1 & 5 \end{vmatrix} = 6$

$i^*_{2,2} = (-1)^{2+2} \cdot \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = -3$; $i^*_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = -3$

$i^*_{3,1} = (-1)^{3+1} \cdot \begin{vmatrix} -2 & 4 \\ 3 & -2 \end{vmatrix} = -8$; $i^*_{3,2} = (-1)^{3+2} \cdot \begin{vmatrix} 1 & 4 \\ -1 & -2 \end{vmatrix} = -2$

$i^*_{3,3} = (-1)^{3+3} \cdot \begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} = 1$

$E_4)$ $I^* = \begin{pmatrix} 13 & 1 & -5 \\ 6 & -3 & -3 \\ -8 & -2 & 1 \end{pmatrix}$

$E_5)$ $I^{-1} = \frac{1}{\det I} \cdot I^* = \frac{1}{-9} \cdot \begin{pmatrix} 13 & 1 & -5 \\ 6 & -3 & -3 \\ -8 & -2 & 1 \end{pmatrix} =$

$I^{-1} = \begin{pmatrix} -13/9 & -1/9 & 5/9 \\ -2/3 & 1/3 & 1/3 \\ 8/9 & 2/9 & -1/9 \end{pmatrix}$

Fie $J = \begin{pmatrix} 4 & 2 & -4 \\ 3 & 2 & -1 \\ -2 & -1 & 5 \end{pmatrix}$, calculați J^{-1}

$E_1)$ $\det J = \begin{vmatrix} 4 & 2 & -4 \\ 3 & 2 & -1 \\ -2 & -1 & 5 \end{vmatrix} = 40+4+12-16-4-30$

$= 56-50 = 6 \neq 0$

$\Rightarrow \exists J^{-1}$

$E_2)$ $J^t = \begin{pmatrix} 4 & 3 & -2 \\ 2 & 2 & -1 \\ -4 & -1 & 5 \end{pmatrix}$

$E_3)$ $j^*_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & -1 \\ -1 & 5 \end{vmatrix} = 10-1 = 9$

$j^*_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 2 & -1 \\ -4 & 5 \end{vmatrix} = -1(10-4) = -6$

$j^*_{13} = (-1)^{1+3} \cdot \begin{vmatrix} 2 & 2 \\ -4 & -1 \end{vmatrix} = -2+8 = 6$

$j^*_{21} = (-1)^{2+1} \cdot \begin{vmatrix} 3 & -2 \\ -1 & 5 \end{vmatrix} = -1(15-2) = -13$

$j^*_{22} = (-1)^{2+2} \cdot \begin{vmatrix} 4 & -2 \\ -4 & 5 \end{vmatrix} = 20-8 = 12$

$j^*_{23} = (-1)^{2+3} \cdot \begin{vmatrix} 4 & 3 \\ -4 & -1 \end{vmatrix} = -1(-4+12) = -8$

$j^*_{31} = (-1)^{3+1} \cdot \begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix} = -3+4 = 1$

$j^*_{32} = (-1)^{3+2} \cdot \begin{vmatrix} 4 & -2 \\ 2 & -1 \end{vmatrix} = -1(-4+4) = 0$

$j^*_{33} = (-1)^{3+3} \cdot \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} = 8-6 = 2$

$E_4)$ $J^* = \begin{pmatrix} 9 & -6 & 6 \\ -13 & 12 & -8 \\ 1 & 0 & 2 \end{pmatrix} \Rightarrow J^{-1} = \frac{1}{\det A} \cdot J^* = \frac{1}{6} \begin{pmatrix} 9 & -6 & 6 \\ -13 & 12 & -8 \\ 1 & 0 & 2 \end{pmatrix} =$

$$\boxed{J^{-1} = \begin{pmatrix} 3/2 & -1 & 1 \\ -13/6 & 2 & -4/3 \\ 1/6 & 0 & 1/3 \end{pmatrix}}$$

Fie $A_4 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$, se cere A_4 are inversă și inversa sa este

$$B = \begin{pmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

E1) se folosește: dacă $A \cdot B = I_4$ și
$B \cdot A = I_4$

$\Rightarrow A$ inversabilă și $A^{-1} = B$

E2) $A_4 \cdot B = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{4}{5} & & \\ -\frac{1}{5} & -\frac{1}{5} & & \\ -\frac{1}{5} & -\frac{1}{5} & & \end{pmatrix}$

$= \begin{pmatrix} \frac{8}{5} - \frac{3}{5} & -\frac{4}{5} + \frac{4}{5} & --- & \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

E3) analog, $B \cdot A_4 = I_4$

E4) $\Rightarrow A_4$ inversabilă și $(A_4)^{-1} = B$

TB4 I16)

Fie $A=\begin{pmatrix}3&2&1\\6&4&2\\9&6&3\end{pmatrix}$ și $B=A+I_3$

Arătați că B este inversabilă și că $B^{-1}=I_3-\frac{1}{11}A$

E_1) Dacă vom arăta că $B\cdot C=I_3$ și că $C\cdot B=I_3$

$\Rightarrow B$ inversabilă și $B^{-1}=C$

E_2) Calculăm $B\cdot\left(I_3-\frac{1}{11}A\right)=(A+I_3)\left(I_3-\frac{1}{11}A\right)=$

$=A-\frac{1}{11}A^2+I_3-\frac{1}{11}A$

E_3) $A^2=\begin{pmatrix}3&2&1\\6&4&2\\9&6&3\end{pmatrix}\begin{pmatrix}3&2&1\\6&4&2\\9&6&3\end{pmatrix}=\begin{pmatrix}30&20&10\\60&40&20\\90&60&30\end{pmatrix}=10A$

E_3) $B\cdot\left(I_3-\frac{1}{11}A\right)=\overset{11)}{A}-\frac{1}{11}\cdot 10A+I_3-\frac{1}{11}A=$

$=I_3+\frac{11A-10A-A}{11}=I_3-O_3=I_3$

E_4) Calculăm $\left(I_3-\frac{1}{11}A\right)B=\left(I_3-\frac{1}{11}A\right)(A+I_3)=$

$=\overset{11)}{A}+I_3-\frac{1}{11}A^2-\frac{1}{11}A\xlongequal{A^2=10A}I_3+\frac{11A-10A-A}{11}=I_3$

E_5) Din $\left.\begin{aligned}B\cdot\left(I_3-\frac{1}{11}A\right)=I_3\\ \left(I_3-\frac{1}{11}A\right)B=I_3\end{aligned}\right\}\Rightarrow B$ inversabilă și $B^{-1}=I_3-\frac{1}{11}A$