FUNCȚII CONSTANTE FOLOSIND DERIVATA

Arătați că:

1) $\sin^2 x + \cos^2 x = 1, \ \forall x \in \mathbb{R}$

2) $\arcsin\sqrt{1-x^2} = \pi - \arccos x, \ \forall x \in (-1;0)$

3) $\arcsin x = \frac{\pi}{2} - \arccos x, \ \forall x \in [-1;1]$

4) $\arccos \frac{1-x^2}{1+x^2} = 2\,\mathrm{arctg}\, x, \ \forall x \geq 0$

5) $\arcsin(3x - 4x^3) = 3\arcsin x, \ \forall x \in \left[-\frac{1}{2}; \frac{1}{2}\right]$

6) $\arcsin \frac{2x}{1+x^2} + 2\,\mathrm{arctg}\, x = \pi, \ \forall x \geq 1$

7) $\mathrm{arctg} \frac{\sqrt{1-x^2}}{1+x} + \frac{1}{2}\arcsin x = \frac{\pi}{4}, \ \forall x \in (-1;1)$

8) $\mathrm{arctg}\, x + \mathrm{arctg}\, \frac{1}{x} = \begin{cases} -\frac{\pi}{2}, & \forall x < 0 \\ \frac{\pi}{2}, & \forall x \in [0;\infty) \end{cases}$

9) $\arcsin \frac{2x}{x^2+1} + 2\,\mathrm{arctg}\, x = \begin{cases} -\pi, & x \leq -1 \\ \pi, & x > 1 \end{cases}$

10) $\mathrm{arctg} \frac{\sqrt{1-x^2}}{x} + \arcsin x = \begin{cases} -\frac{\pi}{2}, & x \in [-1;0) \\ \frac{\pi}{2}, & x \in [0;1] \end{cases}$

Arătați că:

$$\arccos\frac{1-x^2}{1+x^2} = 2\,\mathrm{arctg}\, x, \ \forall x \geq 0$$

E1) Vom arăta că: $\arccos\frac{1-x^2}{1+x^2} - 2\,\mathrm{arctg}\, x = 0$

E2) Fie $f:[0;\infty)\to\mathbb{R}$, $f(x)=\arccos\frac{1-x^2}{1+x^2} - 2\,\mathrm{arctg}\, x$

E3) $$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}\cdot\left(\frac{1-x^2}{1+x^2}\right)' - 2\cdot\frac{1}{1+x^2}$$

$$f'(x) = \frac{-1}{\sqrt{\frac{1+2x^2+x^4-(1-2x^2+x^4)}{(1+x^2)^2}}}\cdot\frac{-2x(1+x^2)-(1-x^2)\cdot 2x}{(1+x^2)^2} - \frac{2}{1+x^2}$$

$$f'(x) = \frac{-|1+x^2|}{\sqrt{4x^2}}\cdot\frac{-2x-\cancel{2x^3}-2x+\cancel{2x^3}}{(1+x^2)^2} - \frac{2}{1+x^2}$$

E4) Dar, $x \geq 0 \Rightarrow f'(x) = \frac{-\cancel{(1+x^2)}}{\cancel{2x}}\cdot\frac{\overset{-2}{\cancel{-4x}}}{(1+x^2)^{\cancel{2}}} - \frac{2}{1+x^2}$

$$f'(x) = \frac{2}{1+x^2} - \frac{2}{1+x^2} \Rightarrow f'(x) = 0$$

E5) $\Rightarrow f$ constantă, $f(x) = k$, $\forall x \in [0;\infty)$

E6) $f(0) = \arccos 1 - 2\,\mathrm{arctg}\, 0 = 0 - 2\cdot 0 = 0$

E7) Din $f(x) = k$, $\forall x \geq 0$ și $f(0) = 0 \Rightarrow$

$\Rightarrow f(x) = 0$, $\forall x \geq 0$

Arătați că: $\arcsin\frac{2x}{x^2+1}+2\,\mathrm{arctg}\,x=\begin{cases}-\pi, & x\le -1\\ \pi, & x>-1\end{cases}$

E1) Cond.: $-1\le\frac{2x}{x^2+1}\le 1 \;\Big|\cdot (x^2+1)>0$

$\underbrace{-x^2-1\le 2x}\underbrace{\le x^2+1}$ $\quad\begin{cases}-x^2-1\le 2x\\ 2x\le x^2+1\end{cases}\Leftrightarrow\begin{cases}(x+1)^2\ge 0\\ (x-1)^2\ge 0\end{cases}$

$\Rightarrow$ adev., $\forall x\in\mathbb{R}\Rightarrow D_f=\mathbb{R}$

E2) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\arcsin\frac{2x}{x^2+1}+2\,\mathrm{arctg}\,x$

E3) $f'(x)=\frac{1}{\sqrt{1-\left(\frac{2x}{x^2+1}\right)^2}}\cdot\left(\frac{2x}{x^2+1}\right)'+2\cdot\frac{1}{1+x^2}$

$f'(x)=\frac{1}{\sqrt{\frac{(x^2+1)^2-4x^2}{(x^2+1)^2}}}\cdot\frac{2(x^2+1)-2x\cdot 2x}{(x^2+1)^2}+\frac{2}{1+x^2}$

$f'(x)=\frac{\cancel{x^2+1}}{\sqrt{x^4+2x^2+1-4x^2}}\cdot\frac{2-2x^2}{(x^2+1)^{\cancel{2}}}+\frac{2}{1+x^2}$

$f'(x)=\frac{2(1-x^2)}{\sqrt{(x^2-1)^2}}\cdot\frac{1}{x^2+1}-\frac{2}{x^2+1}=\frac{2(1-x^2)}{|x^2-1|(x^2+1)}-\frac{2}{x^2+1}$

E4) Caz I: $x^2-1>0\xRightarrow{\text{tabel}} x\in(-\infty;-1)\cup(1;\infty)$

$f'(x)=\frac{2\cancel{(1-x^2)}_{-1}}{\cancel{(x^2-1)}(x^2+1)}+\frac{2}{x^2+1}=\frac{-2}{x^2+1}+\frac{2}{x^2+1}=0$

$\Rightarrow f$ constantă, dar D_f este reuniune de interv. $\Rightarrow f(x)=\begin{cases}k_1, & x\le -1\\ k_2, & x\ge 1\end{cases}$

E5) $f(-1)=\arcsin\frac{-2}{2}+2\,\mathrm{arctg}(-1)=-\frac{\pi}{2}-2\cdot\frac{\pi}{4}=-\pi\Rightarrow f(x)=-\pi,\ \forall x\le -1$

$f(1)=\arcsin\frac{2}{2}+2\,\mathrm{arctg}\,1=\frac{\pi}{2}+2\cdot\frac{\pi}{4}=\frac{\pi}{2}+\frac{\pi}{2}=\pi\Rightarrow f(x)=\pi,\ \forall x\ge 1$

E6) $f(x)=\begin{cases}-\pi, & x\le -1\\ \pi, & x\ge 1\end{cases}$

Arătați că: $\operatorname{arctg} x + \operatorname{arctg} \frac{1}{x} = \begin{cases} -\frac{\pi}{2}, & x < 0 \\ \frac{\pi}{2}, & x > 0 \end{cases}$

E1) $f:(-\infty;0)\cup(0;\infty)\to\mathbb{R},\ f(x)=\operatorname{arctg} x+\operatorname{arctg}\frac{1}{x}$

E2) $f'(x)=\frac{1}{1+x^2}+\left(+\frac{1}{1+\left(\frac{1}{x}\right)^2}\right)\cdot\left(\frac{1}{x}\right)'$

$$f'(x)=\frac{1}{1+x^2}+\left(+\frac{\cancel{x^2}}{x^2+1}\right)\cdot\left(-\frac{1}{\cancel{x^2}}\right)$$

$$f'(x)=\frac{1}{1+x^2}-\frac{1}{x^2+1}\Rightarrow f'(x)=0$$

E3) $\Rightarrow f(x)=k,$

E4) $f:(-\infty;0)\cup(0;\infty)\to\mathbb{R}\Rightarrow f(x)=\begin{cases} k_1, & x<0 \\ k_2, & x>0 \end{cases}$

(pt că D_f este reuniune de intervale)

E5) $f(-1)=\operatorname{arctg}(-1)+\operatorname{arctg}(-1)=-\frac{\pi}{4}-\frac{\pi}{4}=-\frac{2\pi}{4}=-\frac{\pi}{2}$

$$\Rightarrow f(x)=-\frac{\pi}{2},\ \forall x<0$$

$$f(1)=\operatorname{arctg} 1+\operatorname{arctg} 1=\frac{\pi}{4}+\frac{\pi}{4}=\frac{2\pi}{4}=\frac{\pi}{2}$$

$$\Rightarrow f(x)=\frac{\pi}{2},\ \forall x>0$$

E6) $f(x)=\begin{cases} -\frac{\pi}{2}, & x<0 \\ \frac{\pi}{2}, & x>0 \end{cases}$