PRIMITIVELE FUNCȚIILOR TRIGONOMETRICE

Calculați (pe un interval oarecare pe care merge schimbarea de variabilă)

1) $\int \sin x \cdot \cos^2 x \, dx$

2) $\int \sin^2 x \cdot \cos^3 x \, dx$

3) $\int \frac{\cos x}{4+\sin x} \, dx$

4) $\int \frac{\sin x}{4+\cos^2 x} \, dx$

5) $\int \frac{1}{5+\sin x} \, dx$

6) $\int \frac{3}{2+\cos x} \, dx$

~~7)~~ $\int \frac{5}{3+\sin x+\cos x} \, dx$

8) $\int \frac{7}{1+\sin^2 x} \, dx$

9) $\int \frac{\sin x}{\sin x+\cos x} \, dx$

~~10)~~ $\int tg^4 x \, dx$

11) $\int \frac{\cos x}{9+\sin^2 x} \, dx$

12) $\int \frac{\cos^2 x}{5\cos^2 x-2\sin^2 x} \, dx$

13) $\int \frac{2\sin^2 x+3\cos^2 x}{5\cos^2 x+4\sin^2 x} \, dx$

14) $\int \frac{3}{2\cos^2 x-5\sin^2 x} \, dx$

~~15)~~ $\int \frac{1}{2\sin x-\cos x+5} \, dx$

Calculați: $I = \int \sin^2 x \cdot \cos^3 dx$

$E_1)$ $I = \int \sin^2 x \cdot \cos^2 x \cdot \cos x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$

$E_2)$ $t = \sin x \,|\, ()' \Rightarrow dt = \cos x \, dx$

$E_3)$ $I(t) = \int t^2 (1 - t^2) \, dt = \int (t^2 - t^4) \, dt$

$E_4)$ $I(t) = \frac{t^3}{3} - \frac{t^5}{5} + \mathcal{C}$

$E_5)$ $\boxed{I(x) = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + \mathcal{C}}$

Calculați:

$$I=\int \frac{5}{3+\sin x+\cos x}\,dx$$

$E_1)$ folosim $\sin x=\dfrac{2\,tg\frac{x}{2}}{1+tg^2\frac{x}{2}}$ și $\cos x=\dfrac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}$

$E_2)$ $$I=\int \frac{5}{3+\frac{2\,tg\frac{x}{2}}{1+tg^2\frac{x}{2}}+\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}}\,dx$$

$E_3)$ not $tg\frac{x}{2}=t \Rightarrow \frac{x}{2}=arctg\,t \Rightarrow x=2\,arctg\,t\,/()'$

$$\Rightarrow dx=2\cdot\frac{1}{1+t^2}\,dt$$

$E_4)$ $$I(t)=\int \frac{5}{{}^{1+t^2)}3+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt$$

$$I(t)=\int \frac{5(\cancel{1+t^2})}{3+3t^2+2t+1-t^2}\cdot\frac{2}{\cancel{1+t^2}}\,dt$$

$$I(t)=\int \frac{10}{2t^2+2t+4}\,dt=\int\frac{5}{t^2+t+2}\,dt$$

$E_5)$ forma canonică $ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a}$

$$\Rightarrow I(t)=5\int\frac{1}{1\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}\,dt,\ \text{not } t+\frac{1}{2}=u\,/()' \Rightarrow dt=du$$

$$I(u)=5\int\frac{1}{u^2+\frac{3}{4}}\,du=5\cdot\frac{1}{\sqrt{\frac{3}{4}}}\,arctg\frac{u}{\sqrt{\frac{3}{4}}}+\mathcal{C}$$

$$\Rightarrow I(t)=\frac{10}{\sqrt3}\,arctg\frac{2u}{\sqrt3}+\mathcal{C}\Rightarrow I(x)=\frac{10\sqrt3}{3}\,arctg$$

Calculați: $I=\int \frac{\sin x}{\sin x+\cos x}\,dx$

$E_1)\ \sin x=\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}},\quad \cos x=\frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}$

$E_2)\ I(x)=\int \frac{\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}}{\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}+\frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}}\,dx,\quad \operatorname{tg}\frac{x}{2}=t \Rightarrow \frac{x}{2}=\operatorname{arctg} t$

$\Rightarrow x=2\operatorname{arctg} t \,|\,()'$

$\Rightarrow dx=\frac{2}{1+t^2}\,dt$

$E_3)\ I(t)=\int \frac{\frac{2t}{\cancel{1+t^2}}}{\frac{2t+1-t^2}{\cancel{1+t^2}}}\cdot\frac{2}{1+t^2}\,dt=4\int \frac{t}{(1+t^2)(2t+1-t^2)}\,dt$

$E_4)\ \frac{t}{(t^2+1)(-t^2+2t+1)}=\frac{At+B}{t^2+1}+\frac{Ct+\Delta}{-t^2+2t+1}=\frac{(At+B)(-t^2+2t+1)+(Ct+\Delta)(t^2+1)}{(t^2+1)(-t^2+2t+1)}$

$=\frac{t^3(C-A)+t^2(2A-B+\Delta)+t(A+2B+C)+B+\Delta}{(t^2+1)(-t^2+2t+1)}$

Egalăm coef $\Rightarrow \begin{cases} C-A=0\\ 2A-B+\Delta=0\\ A+2B+C=1\\ B+\Delta=0\end{cases}$ $\Rightarrow A=\frac{1}{3},\ B=\frac{1}{3},\ C=\frac{1}{3},\ \Delta=-\frac{1}{3}$

$E_5)\ I(t)=4\left(\int\frac{\frac{1}{3}t+\frac{1}{3}}{t^2+1}\,dt+\int\frac{\frac{1}{3}t-\frac{1}{3}}{-t^2+2t+1}\,dt\right)$

$I(t)=\frac{4}{3}\left(\int\frac{t+1}{t^2+1}\,dt+\int\frac{t-1}{-t^2+2t+1}\,dt\right)=$

$=\frac{4}{3}\left(\frac{1}{2}\int\frac{2t}{t^2+1}\,dt+\int\frac{1}{t^2+1}\,dt-\frac{1}{2}\int\frac{-2t+2}{-t^2+2t+1}\,dt\right)$

$=\frac{4}{3}\left(\frac{1}{2}\ln|t^2+1|+\frac{1}{1}\operatorname{arctg}\frac{t}{1}-\frac{1}{2}\ln|-t^2+2t+1|\right)+\mathcal{C}$

$E_6)\ I(x)=\frac{4}{3}\left(\frac{1}{2}\ln\left|\operatorname{tg}^2\frac{x}{2}+1\right|+\frac{x}{2}-\frac{1}{2}\ln\left|-\operatorname{tg}^2\frac{x}{2}+2\operatorname{tg}\frac{x}{2}+1\right|\right)+\mathcal{C}$

Calculați: $I=\int tg^4 x\,dx$

E1) $t = tg\,x \Leftrightarrow x = arctg\,t \;\; ||()' \Rightarrow dx = \frac{1}{1+t^2}dt$

E2) $I(t) = \int t^4 \cdot \frac{1}{t^2+1}\,dt = \int \frac{t^4}{t^2+1}\,dt$

E3)
$$\begin{array}{l|l} t^4 & t^2+1 \\ \hline -t^4-t^2 & t^2-1 \\ \quad\; -t^2 & \\ \quad\;\; t^2+1 & \\ \qquad\;\; 1 & \end{array}$$

$\Rightarrow I(t) = \int (t^2-1)\,dt + \int \frac{1}{t^2+1}\,dt$

E4) $I(t) = \frac{t^3}{3} - t + \frac{1}{1}\,arctg\,\frac{t}{1} + C$

E5) $I(x) = \frac{tg^3 x}{3} - tg\,x + arctg(tg\,x) + C$

$$\boxed{I(x) = \frac{tg^3 x}{3} - tg\,x + x + C}$$

Calculați: $\displaystyle\int \frac{1}{2\sin x-\cos x+5}\,dx$

$E_1)$ $\displaystyle I(x)=\int \frac{1}{2\cdot\frac{2\,tg\frac{x}{2}}{1+tg^2\frac{x}{2}}-\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}+5}\,dx$

$E_2)$ $t=tg\frac{x}{2}\Rightarrow \frac{x}{2}=arctg\, t\Rightarrow x=2\,arctg\, t\ |()'$

$dx=2\cdot\frac{1}{1+t^2}\,dt$

$E_3)$ $\displaystyle I(t)=\int \frac{1}{\frac{4t}{1+t^2}-\frac{1-t^2}{1+t^2}+5}\cdot\frac{2}{1+t^2}\,dt=$

$\displaystyle =\int \frac{1}{\frac{4t-1+t^2+5+5t^2}{\cancel{1+t^2}}}\cdot\frac{2}{\cancel{1+t^2}}\,dt=$

$\displaystyle =\int \frac{\cancel{2}}{6t^2+4t+4}\,dt=\int\frac{1}{3t^2+2t+2}\,dt$

$E_4)$ $\displaystyle I(t)=\int\frac{1}{3\left(t+\frac{2}{6}\right)^2+\frac{20}{12}}\,dt=\int\frac{dt}{3\left(t+\frac{1}{3}\right)^2+\frac{5}{3}}$

$E_5)$ not $t+\frac{1}{3}=u\ |()'\Rightarrow dt=du$

$\displaystyle I(u)=\frac{1}{3}\int\frac{1}{u^2+\frac{5}{9}}\,du=\frac{1}{\cancel{3}}\cdot\frac{1}{\frac{\sqrt{5}}{\cancel{3}}}\,arctg\frac{u}{\sqrt{\frac{5}{9}}}+\mathcal{C}$

$\displaystyle \Rightarrow I(t)=\frac{1}{\sqrt{5}}\,arctg\frac{3\left(t+\frac{1}{3}\right)}{\sqrt{5}}+\mathcal{C}=\frac{1}{\sqrt{5}}\,arctg\frac{3\,tg\frac{x}{2}+1}{\sqrt{5}}+\mathcal{C}$