CALCULUL PRIMITIVELOR FOLOSIND METODA INTEGRĂRII PRIN PĂRȚI

Calculați:

1) $\int x \ln x \, dx$ și $\int \ln x \, dx$

2) $\int x e^{-x} dx$ și $\int x^2 e^{x} dx$

3) $\int \frac{\ln x}{x^2} dx$ și $\int x e^{2x} dx$

4) $\int x \arcsin x \, dx$ și $\int x \operatorname{arctg} x \, dx$

5) $\int \arccos x \, dx$ și $\int x \operatorname{arctg} \frac{1}{x} dx$

6) $\int e^{x} \sin x \, dx$ și $\int e^{x} \cos x \, dx$

7) $\int e^{2x} \sin 3x \, dx$ și $\int x \sin^2 x \, dx$

8) $\int x^2 \cdot \ln x \, dx$ și $\int x^2 \cos 2x \, dx$

9) $\int x^2 \cdot 5^{3x} dx$ și $\int x \cos 5x \, dx$

10) $\int \ln(x^2+1) dx$ și $\int \sin(\ln x) dx$

11) $\int x^3 \ln^2 x \, dx$ și $\int e^{x} \cos^2 x \, dx$

12) $\int \frac{x \operatorname{arctg} x}{\sqrt{1+x^2}} dx$ și $\int (x^2+x) e^{3x} dx$

13) $\int x \operatorname{arctg} x^2 \, dx$ și $\int x e^{x} \sin x \, dx$

14) $\int x e^{x} \cos x \, dx$ și $\int \frac{x \sin x}{\cos^2 x} dx$

15) $\int \frac{1}{\cos^6 x} dx \left(\frac{1}{\cos^6 x} = \frac{1}{\cos^4 x} (\operatorname{tg} x)' \right)$ și $\int \frac{dx}{\sin^6 x}$

16) $\int x \sin x \cdot e^{x} dx$

Calculați: a) $\int x \ln x \, dx$, $x \in (0;\infty)$

b) $\int \ln x \, dx$, $x \in (0;\infty)$

a)

E_1) Integrare prin părți,

$$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$$

$$g'(x) = x \Rightarrow g(x) = \frac{x^2}{2}$$

$$E_2) \; I = \frac{x^2}{2} \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx$$

$$\underline{I = \frac{x^2}{2} \ln x - x + C}$$

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b)

E_1) pt $F(x) = \int \ln x \, dx$

$$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$$

$$g'(x) = 1 \Rightarrow g(x) = x$$

$$E_2) \; F(x) = x \ln x - \int \frac{1}{\cancel{x}} \cdot \cancel{x} \, dx$$

$$F(x) = x \ln x - x + C$$

Calculați: a) $\int \arccos x \, dx$

b) $\int x \operatorname{arctg} \frac{1}{x} \, dx$

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a) $I = \int \arccos x \, dx$

$f = \arccos x \Rightarrow f' = -\frac{1}{\sqrt{1-x^2}}$

$g' = 1 \Rightarrow g = x$

$I = x \arccos x + \int \frac{x}{\sqrt{1-x^2}} \, dx$

$$\boxed{I = x \arccos x - \sqrt{1-x^2} + C}$$

b) $I = \int x \cdot \operatorname{arctg} \frac{1}{x} \, dx$

$f = \operatorname{arctg} \frac{1}{x} \Rightarrow f' = \frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{\frac{x^2+1}{\cancel{x^2}}} \left(-\frac{1}{\cancel{x^2}}\right)$

$\Rightarrow f = \operatorname{arctg} \frac{1}{x} \Rightarrow f' = -\frac{1}{x^2+1}$

$g' = x \Rightarrow g = \frac{x^2}{2}$

$I = \frac{x^2}{2} \operatorname{arctg} \frac{1}{x} + \frac{1}{2} \int \frac{x^2}{x^2+1} \, dx$

$I = \frac{x^2}{2} \operatorname{arctg} \frac{1}{x} + \frac{1}{2} \int \frac{x^2+1-1}{x^2+1} \, dx$

$I = \frac{x^2}{2} \operatorname{arctg} \frac{1}{x} + \frac{1}{2} \left(\int 1 \, dx - \int \frac{1}{x^2+1} \, dx \right)$

$$\boxed{I = \frac{x^2}{2} \operatorname{arctg} \frac{1}{x} + \frac{1}{2}\left(x - \frac{1}{1} \operatorname{arctg} \frac{x}{1}\right) + C}$$

Calculați: a) $\int e^{2x}\sin 3x\,dx$

b) $\int x\sin^2 x\,dx$

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a) $I=\int e^{2x}\sin 3x\,dx$

$f=\sin 3x \Rightarrow f'=(\cos 3x)\cdot 3$

$g'=e^{2x} \Rightarrow g=\int e^{2x}dx=\frac{e^{2x}}{2}$

$I=\frac{e^{2x}}{2}\sin 3x-\frac{3}{2}\underbrace{\int e^{2x}\cdot\cos 3x\,dx}_{I_1}$

pt I_1, $f=\cos 3x \Rightarrow f'=(-\sin 3x)\cdot 3$

$g'=e^{2x} \Rightarrow g=\frac{e^{2x}}{2}$

$I_1=\frac{e^{2x}}{2}\cos 3x+\frac{3}{2}\underbrace{\int e^{2x}\sin 3x\,dx}_{I}$

înlocuind $\Rightarrow I=\frac{e^{2x}}{2}\sin 3x-\frac{3}{2}\left(\frac{e^{2x}}{2}\cos 3x+\frac{3}{2}I\right)$

$I=\frac{e^{2x}}{2}\sin 3x-\frac{3e^{2x}}{4}\cos 3x-\frac{9}{4}I \;\Big|\cdot 4$

$4I=2e^{2x}\sin 3x-3e^{2x}\cos 3x-9I$

$I=\frac{1}{13}\left(2e^{2x}\sin 3x-3e^{2x}\cos 3x\right)+C$

b) $I=\int x\sin^2 x\,dx=\int x\cdot\frac{1-\cos 2x}{2}dx=\frac{1}{2}\left(\int x\,dx-\underbrace{\int x\cos 2x\,dx}_{I_1}\right)$

pt I_1, $f=x\Rightarrow f'=1$

$g'=\cos 2x\Rightarrow g=\frac{\sin 2x}{2}$

$\Rightarrow I_1=x\cdot\frac{\sin 2x}{2}-\frac{1}{2}\int\sin 2x\,dx$

$\Rightarrow I_1=x\cdot\frac{\sin 2x}{2}-\frac{1}{2}\cdot\left(-\frac{\cos 2x}{2}\right)+C$

$$\boxed{I=\frac{1}{2}\left(\frac{x^2}{2}-x\cdot\frac{\sin 2x}{2}-\frac{1}{4}\cdot\cos 2x\right)+C}$$

Calculați: a) $\int x^2 \ln x \, dx$

b) $\int x^2 \cos 2x \, dx$

a)

$E_1)$ $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$

$g'(x) = x^2 \Rightarrow g(x) = \frac{x^3}{3}$

$E_2)$ $I = \frac{x^3}{3} \cdot \ln x - \int \frac{1}{\cancel{x}} \cdot \frac{x^{\cancel{3}\,2}}{3} dx$

$= \frac{x^3}{3} \cdot \ln x - \frac{1}{3} \int x^2 dx$

$E_3)$ $I = \frac{x^3}{3} \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + C$

$\underline{I = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C}$

b) pt $I = \int x^2 \cos 2x \, dx$.

$f(x) = x^2 \rightarrow f'(x) = 2x$

$g'(x) = \cos 2x \rightarrow g(x) = \frac{\sin 2x}{2}$

$I = x^2 \cdot \frac{\sin 2x}{2} - \int \cancel{2}x \cdot \frac{\sin 2x}{\cancel{2}} dx = x^2 \cdot \frac{\sin 2x}{2} - \underbrace{\int x \sin 2x \, dx}_{I_1}$

pt I_1, $f(x) = x \rightarrow f'(x) = 1$

$g'(x) = \sin 2x \rightarrow g(x) = -\frac{\cos 2x}{2}$

$I_1 = -x \cdot \frac{\cos 2x}{2} + \frac{1}{2} \int \cos 2x \, dx = -x \cdot \frac{\cos 2x}{2} + \frac{1}{2} \frac{\sin 2x}{2} + C$

$I = x^2 \cdot \frac{\sin 2x}{2} - \left(-x \cdot \frac{\cos 2x}{2} + \frac{\sin 4x}{4}\right) + C = \frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 4x}{4} + C$

Calculați: a) $\int x^2 \cdot 5^{3x}\,dx$

b) $\int x\cos 5x\,dx$

a)

E1) $I = \int x^2 \cdot 5^{3x}\,dx$

$f = x^2 \Rightarrow f' = 2x$

$g' = 5^{3x} \Rightarrow g = \int 5^{3x}\,dx = \frac{5^{3x}}{3\ln 5}$

$I = x^2 \cdot \frac{5^{3x}}{3\ln 5} - \frac{2}{3\ln 5}\underbrace{\int x\cdot 5^{3x}\,dx}_{I_1}$

pt I_1, $f = x \Rightarrow f' = 1$

$g' = 5^{3x} \Rightarrow g = \frac{5^{3x}}{3\ln 5}$

$I_1 = x\cdot\frac{5^{3x}}{3\ln 5} - \frac{1}{3\ln 5}\int 5^{3x}\,dx$

$I_1 = x\cdot\frac{5^{3x}}{3\ln 5} - \frac{1}{3\ln 5}\cdot\frac{5^{3x}}{3\ln 5} + C$

$I = x^2\cdot\frac{5^{3x}}{3\ln 5} - \frac{2}{3\ln 5}\left(x\cdot\frac{5^{3x}}{3\ln 5} - \frac{5^{3x}}{9\ln^2 5}\right) + C$

b) pt $I = \int x\cos 5x\,dx$

$f(x) = x \Rightarrow f'(x) = 1$

$g'(x) = \cos 5x \Rightarrow g(x) = \frac{\sin 5x}{5}$

$I = \frac{\sin 5x}{5}\cdot x - \frac{1}{5}\int \sin 5x\,dx$

$I = \frac{1}{5}x\sin 5x + \frac{1}{5}\,\frac{\cos 5x}{5} + C$

Calculați $F(x)=\int x\sin x\cdot e^x\,dx$

$E_1)\ f(x)=x\sin x \Rightarrow f'(x)=\sin x+x\cos x$

$g'(x)=e^x \Rightarrow g(x)=e^x$

$E_2)\ F(x)=x\sin x\cdot e^x-\int(\sin x+x\cos x)e^x$

$F(x)=x\sin x\, e^x-\Big(\int\sin x\cdot e^x\,dx+\underbrace{\int x\cos x\, e^x\,dx}_{I_1}\Big)$

$E_3)$ pt $I_1 \Rightarrow f(x)=x\cos x \Rightarrow f'(x)=\cos x-x\sin x$

$g'(x)=e^x \Rightarrow g(x)=e^x$

$E_4)\ I_1=x\cos x\, e^x-\int(\cos x-x\sin x)e^x\,dx=$

$=x\cos x\, e^x-\int\cos x\, e^x+\underbrace{\int x\sin x\, e^x\,dx}_{F(x)}$

$E_5)\ F(x)=x\sin x\, e^x-\Big(\int\sin x\, e^x\,dx+x\cos x\, e^x-\int\cos x\, e^x+F(x)\Big)$

$F(x)=x\sin x\, e^x-\int\sin x\, e^x\,dx-x\cos x\, e^x+\int\cos x\, e^x-F(x)$

$F(x)=\frac{1}{2}\Big[x\sin x\, e^x-x\cos x\, e^x-\underbrace{\int\sin x\, e^x\,dx}_{I_2}+\underbrace{\int\cos x\, e^x\,dx}_{I_4}\Big]$

$E_6)$ pt $I_2 \Rightarrow f(x)=\sin x \Rightarrow f'(x)=\cos x$

$g'(x)=e^x \Rightarrow g(x)=e^x$

$I_2=e^x\sin x-\underbrace{\int e^x\cos x\,dx}_{I_3}$, pt $I_3 \Rightarrow f=\cos x \Rightarrow f'=-\sin x$; $g'=e^x \Rightarrow g=e^x$

$I_3=e^x\cos x+\int e^x\sin x\,dx \Rightarrow I_2=e^x\sin x-(e^x\cos x+I_2)$

$\Rightarrow \boxed{I_2=\frac{1}{2}\left[e^x\sin x-e^x\cos x\right]+\mathcal{C}}$

$E_7)$ pt $I_4=\int\cos x\, e^x\,dx \Rightarrow f=\cos x \Rightarrow f'(x)=-\sin x$

$g'=e^x \Rightarrow g(x)=e^x$

$I_4=e^x\cos x+\int e^x\sin x\,dx=e^x\cos x+\frac{1}{2}\left[e^x\sin x-e^x\cos x\right]+\mathcal{C}=\frac{1}{2}\left[e^x\sin x+e^x\cos x\right]$

$E_8)\ F(x)=\frac{1}{2}\Big[x e^x(\sin x-\cos x)-\frac{1}{2}(e^x\sin x-e^x\cos x)+\frac{1}{2}(e^x\sin x+e^x\cos x)\Big]+\mathcal{C}$

$=\frac{1}{2}\left[x e^x(\sin x-\cos x)+e^x\cos x\right]+\mathcal{C}$