CALCULAREA LIMITELOR DIN $n \cdot I_n$

Calculați $\lim_{n\to\infty} n I_n$ în situațiile:

1) $I_n = \int_0^1 \frac{x^n}{4x+5}\,dx$

2) $I_n = \int_0^1 \frac{x^n}{x^2+3x+5}\,dx$

3) $I_n = \int_0^1 \frac{x^n}{x^2+3x+5}\,dx$

4) $I_n = \int_0^1 \frac{x^n}{3x+8}\,dx$

5) $I_n = \int_0^1 \frac{x^n}{x^2+5x+6}\,dx$

6) $I_n = \int_0^1 \frac{x^n}{x^2+x+3}\,dx$

7) $I_n = \int_0^1 \frac{x^n}{x^2+1}\,dx$

8) $I_n = \int_0^1 \frac{x^n}{2x^2+3x+1}\,dx$

9) $I_n = \int_0^1 \frac{x^n}{5x+1}\,dx$

10) $I_n = \int_0^1 \frac{x^n}{3x^2+2x+5}\,dx$

Fie $I_n=\int_0^1 \frac{x^n}{4x+5}\,dx$

Determinați $\lim_{n\to\infty} nI_n$

$E_1)$ studiem monotonia lui I_n:

$$I_{n+1}-I_n=\int_0^1 \frac{x^{n+1}}{4x+5}dx-\int_0^1 \frac{x^n}{4x+5}dx=$$

$$=\int_0^1 \frac{x^n(x-1)}{4x+5}dx$$

Dar, $x\in[0;1]\Rightarrow\begin{cases} x^n\geq 0\\ x-1\leq 0\\ 4x+5>0\end{cases}\Rightarrow \frac{x^n(x-1)}{4x+5}\leq 0$

$E_2) \Rightarrow I_{n+1}-I_n\leq 0\Rightarrow I_{n+1}\leq I_n$

$E_3)$ Calculăm $4I_{n+1}+5I_n=4\int_0^1\frac{x^{n+1}}{4x+5}dx+5\int_0^1\frac{x^n}{4x+5}dx=$

$$=\int_0^1 \frac{4x^{n+1}+5x^n}{4x+5}dx=\int_0^1\frac{x^n\cancel{(4x+5)}}{\cancel{4x+5}}dx=$$

$$=\int_0^1 x^n dx=\frac{x^{n+1}}{n+1}\Big|_0^1=\frac{1}{n+1}-0=\frac{1}{n+1}$$

$E_4)$ $I_{n+1}\leq I_n \;|\cdot 4\Rightarrow 4I_{n+1}\leq 4I_n\;|+5I_n$

$\Rightarrow 4I_{n+1}+5I_n\leq 9I_n\Rightarrow\frac{1}{n+1}\leq 9I_n\;|\cdot\frac{n}{9}\Rightarrow \boxed{\frac{n}{9(n+1)}\leq nI_n}$

$E_5)$ $I_{n+1}\leq I_n\;|\cdot 5\Rightarrow 5I_{n+1}\leq 5I_n\;|+4I_{n+1}\Rightarrow 9I_{n+1}\leq 5I_n+4I_{n+1}$

$\Rightarrow 9I_{n+1}\leq\frac{1}{n+1}\Rightarrow 9I_n\leq\frac{1}{n}\;|\cdot\frac{n}{9}\Rightarrow\boxed{nI_n\leq\frac{1}{9}}$

$E_6)$ $\frac{n}{9(n+1)}\leq nI_n\leq\frac{1}{9}$ și din cleşte $\Rightarrow nI_n\to\frac{1}{9}$

Fie $I_n = \int_0^1 \frac{x^n}{x^2+3x+5}\,dx$

Calculați $\lim\limits_{n\to\infty} n\,I_n$

$E_1)$ Fie $\int_0^1 \frac{x^n(x^2+3x+5)}{x^2+3x+5}\,dx = \int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$

$E_2)$ Dar, $\int_0^1 \frac{x^n(x^2+3x+5)}{x^2+3x+5}\,dx = \int_0^1 \frac{x^{n+2}+3x^{n+1}+5x^n}{x^2+3x+5}\,dx =$

$= \int_0^1 \frac{x^{n+2}}{x^2+3x+5}\,dx + 3\int_0^1 \frac{x^{n+1}}{x^2+3x+5}\,dx + 5\int_0^1 \frac{x^n}{x^2+3x+5}\,dx$

$= I_{n+2} + 3I_{n+1} + 5I_n$

$E_3) \Rightarrow I_{n+2} + 3I_{n+1} + 5I_n = \frac{1}{n+1}$

$E_4)\ I_{n+1} - I_n = \int_0^1 \frac{x^n(x-1)}{x^2+3x+5}\,dx \le 0 \Rightarrow I_{n+1} \le I_n$

$E_3)\ I_{n+2} + 3I_{n+1} + 5I_n \le I_n + 3I_n + 5I_n = 9I_n$

$\Rightarrow \frac{1}{n+1} \le 9I_n \Rightarrow \boxed{I_n \ge \frac{1}{9(n+1)}}$

$E_4)\ I_{n+2} + 3I_{n+1} + 5I_n \ge I_{n+2} + 3I_{n+2} + 5I_{n+2} = 9I_{n+2}$

$\Rightarrow \frac{1}{n+1} \ge 9I_{n+2} \Rightarrow I_{n+2} \le \frac{1}{9(n+1)} \Rightarrow \boxed{I_n \le \frac{1}{9(n-1)}}$

$E_5)\ \frac{1}{9(n+1)} \le I_n \le \frac{1}{9(n-1)}\ \Big|\cdot n \Rightarrow \underbrace{\frac{n}{9n+9}}_{\to \frac{1}{9}} \le nI_n \le \underbrace{\frac{n}{9n-9}}_{\to\frac{1}{9}}$

$- I \to \frac{1}{\ldots}$

Calculați $\lim_{n\to\infty} n\cdot I_n$ pentru $I_n=\int_0^1 \frac{x^n}{x^2+3x+5}\,dx$

E1) caut relație de recurență pt I_n

$$I=\int_0^1 \frac{x^n(x^2+3x+5)}{x^2+3x+5}\,dx=\int_0^1 \frac{x^{n+2}+3x^{n+1}+5x^n}{x^2+3x+5}\,dx=$$

$$=\int_0^1 \frac{x^{n+2}}{x^2+3x+5}\,dx+3\int_0^1 \frac{x^{n+1}}{x^2+3x+5}\,dx+5\int_0^1 \frac{x^n}{x^2+3x+5}\,dx$$

$$\Rightarrow I=I_{n+2}+3I_{n+1}+5I_n$$

Dar, $I=\int_0^1 \frac{x^n(\cancel{x^2+3x+5})}{\cancel{x^2+3x+5}}\,dx=\frac{x^{n+1}}{n+1}\Big|_0^1=\frac{1}{n+1}$

$$\Rightarrow \boxed{I_{n+2}+3I_{n+1}+5I_n=\frac{1}{n+1}}$$

E2) monotonia lui I_n:

$$I_{n+1}-I_n=\int_0^1 \frac{x^{n+1}}{x^2+3x+5}\,dx-\int_0^1 \frac{x^n}{x^2+3x+5}\,dx=$$

$$=\int_0^1 \frac{x^n(x-1)}{x^2+3x+5}\,dx$$

Cum $x\in[0;1]\Rightarrow x-1\le 0\Rightarrow \frac{x^n(x-1)}{x^2+3x+5}\le 0 \Big| \int_0^1 \ldots dx \Rightarrow$

$$I_{n+1}-I_n\le 0\Rightarrow \boxed{I_{n+1}\le I_n}\Rightarrow I_{n+2}\le I_{n+1}\le I_n$$

E3) din $I_{n+2}\le I_{n+1}\le I_n\Rightarrow I_{n+2}+3I_{n+1}+5I_n\le 9I_n$

$$\Rightarrow \frac{1}{n+1}\le 9I_n\Rightarrow \frac{1}{9(n+1)}\le I_n \Big|\cdot n\Rightarrow \boxed{\frac{n}{9(n+1)}\le nI_n}$$

E4) din $I_{n+2}\le I_{n+1}\le I_n\Rightarrow I_{n+2}+3I_{n+1}+5I_n\ge 9I_{n+2}$

$$\Rightarrow \frac{1}{n+1}\ge 9I_{n+2}\Rightarrow \frac{1}{n-1}\ge 9I_n \Big|\cdot \frac{n}{9}\Rightarrow \boxed{\frac{n}{9(n-1)}\ge nI_n}$$

E5) $\frac{n}{9(n+1)}\le nI_n\le \frac{n}{9(n-1)}\Rightarrow \lim_{n\to\infty} nI_n=\frac{1}{9}$

Fie $I_n = \int_0^1 \frac{x^n}{3x+8}\,dx$, calculați

$\lim\limits_{n\to\infty} n I_n$

E1) stabilim o relație de recurență direct folosind că

$$\int_0^1 \frac{x^n(3x+8)}{3x+8}\,dx = \int_0^1 \frac{3x^{n+1}}{3x+8}\,dx + \int_0^1 \frac{8x^n}{3x+8}\,dx$$

$$\Rightarrow \int_0^1 x^n\,dx = 3I_{n+1} + 8I_n \Rightarrow \boxed{3I_{n+1} + 8I_n = \frac{1}{n+1}}$$

E2) stabilim monot lui I_n:

$$I_{n+1} - I_n = \int_0^1 \frac{x^{n+1}}{3x+8}\,dx - \int_0^1 \frac{x^n}{3x+8}\,dx = \int_0^1 \frac{x^n(x-1)}{3x+8}\,dx$$

și cum $x \in [0;1] \Rightarrow \frac{x^n(x-1)}{3x+8} \le 0 \Rightarrow I_{n+1} \le I_n \Rightarrow \boxed{I_n \searrow}$

E3) $I_n \searrow \Rightarrow I_{n+1} \le I_n \Rightarrow 11 I_{n+1} \le 3I_{n+1} + 8I_n \le 11 I_n$

E4) Dar, $3I_{n+1} + 8I_n = \frac{1}{n+1}$

$$\Rightarrow 11 I_{n+1} \le \frac{1}{n+1} \le 11 I_n \Rightarrow I_{n+1} \le \frac{1}{11(n+1)} \le I_n$$

E5) Din $\frac{1}{11(n+1)} \le I_n \,\big|\cdot n \Rightarrow \frac{n}{11(n+1)} \le n I_n$ (1)

Din $I_{n+1} \le \frac{1}{11(n+1)} \Rightarrow I_n \le \frac{1}{11n} \,\big|\cdot n \Rightarrow n I_n \le \frac{1}{11}$ (2)

E6) Din (1) + (2) $\Rightarrow \frac{n}{11(n+1)} \le n I_n \le \frac{1}{11}$

$$\searrow \frac{1}{11} \qquad \frac{1}{11} \swarrow$$

$$\Rightarrow n I_n \to \frac{1}{11} \Rightarrow \boxed{\lim_n n I_n = \frac{1}{11}}$$

Fie $I_n=\int_0^1 \frac{x^n}{x^2+1}dx$, se cere

$\lim_n nI_n=?$

$E_1)$ $I_{n+2}+I_n=\int_0^1 \frac{x^{n+2}}{x^2+1}dx+\int_0^1 \frac{x^n}{x^2+1}dx=$

$=\int_0^1 \frac{x^n\cancel{(x^2+1)}}{\cancel{x^2+1}}dx=\int_0^1 x^n dx=\frac{x^{n+1}}{n+1}\Big|_0^1$

$\Rightarrow I_{n+2}+I_n=\frac{1}{n+1}$

$E_2)$ monotonia; $I_{n+1}-I_n=\int_0^1 \frac{x^{n+1}}{x^2+1}dx-\int_0^1 \frac{x^n}{x^2+1}dx$

$=\int_0^1 \frac{x^n(x-1)}{x^2+1}dx$

Dar, $x\in[0;1]\Rightarrow\begin{cases} x^n\geq 0\\ x-1\leq 0\\ x^2+1>0\end{cases}\Rightarrow \frac{x^n(x-1)}{x^2+1}\geq 0$

$\Rightarrow I_{n+1}-I_n\geq 0\Rightarrow I_{n+1}\geq I_n\Rightarrow I_{n+2}\geq I_{n+1}\geq I_n$

$E_3)$ Asadar, $I_{n+2}\geq I_n \,|+I_n\Rightarrow I_{n+2}+I_n\geq 2I_n$

$\Rightarrow \frac{1}{n+1}\geq 2I_n \,|\cdot\frac{n}{2}\Rightarrow \boxed{\frac{n}{2(n+1)}\geq nI_n}$

$E_4)$ $I_{n+2}\geq I_n \,|+I_{n+2}\Rightarrow 2I_{n+2}\geq I_{n+2}+I_n\Rightarrow$

$2I_{n+2}\geq\frac{1}{n+1}\xRightarrow{n\to n-2} 2I_n\geq\frac{1}{n-1}\,|\cdot\frac{n}{2}\Rightarrow\boxed{nI_n\geq\frac{n}{2(n-1)}}$

$E_5)$ $\underbrace{\frac{n}{2(n+1)}}_{\to 1/2}\geq nI_n\geq\underbrace{\frac{n}{2(n-1)}}_{\to 1/2}\Rightarrow\boxed{\lim_n nI_n=\frac{1}{2}}$

Fie $I_n = \int_0^1 \frac{x^n}{3x^2+2x+5}\,dx$, calculați

$\lim\limits_{n\to\infty} n I_n$

E_1) monotonia: $I_{n+1} - I_n = \int_0^1 \frac{x^{n+1}}{3x^2+2x+5}\,dx - \int_0^1 \frac{x^n\,dx}{3x^2+2x+5}$

$= \int_0^1 \frac{x^n(x-1)}{3x^2+2x+5}\,dx$. Dar, $x \in [0;1] \Rightarrow x^n(x-1) \leq 0$

$\Rightarrow \frac{x^n(x-1)}{3x^2+2x+5} \leq 0 \,\Big|\, \int_0^1 \ldots dx \Rightarrow I_{n+1} - I_n \leq 0$

$\Rightarrow I_{n+1} \leq I_n \Rightarrow I_n \searrow$

E_2) deducem o relație de recurență astfel:

$I = \int_0^1 \frac{x^n(3x^2+2x+5)}{3x^2+2x+5}\,dx = \int_0^1 \frac{3x^{n+2}+2x^{n+1}+5x^n}{3x^2+2x+5}\,dx =$

$= 3I_{n+2} + 2I_{n+1} + 5I_n$

Dar, $I = \int_0^1 \frac{x^n\cancel{(3x^2+2x+5)}}{\cancel{3x^2+2x+5}}\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$

$\Rightarrow \boxed{3I_{n+2} + 2I_{n+1} + 5I_n = \frac{1}{n+1}}$

E_3) din $I_{n+1} \leq I_n \Rightarrow I_{n+2} \leq I_{n+1} \leq I_n$

$\Rightarrow 10\,I_{n+2} \leq 3I_{n+2} + 2I_{n+1} + 5I_n \leq 10 I_n$

$\Rightarrow 10\,I_{n+2} \leq \frac{1}{n+1} \leq 10 I_n$

Dar, din $10 I_{n+2} \leq \frac{1}{n+1} \Rightarrow 10 I_n \leq \frac{1}{n-1}$

E_4) înlocuim în (1) $\Rightarrow \frac{1}{n+1} \leq 10 I_n \leq \frac{1}{n-1} \Big| \cdot \frac{n}{10}$

$\Rightarrow \underbrace{\frac{n}{10(n+1)}}_{\to 1} \leq n I_n \leq \underbrace{\frac{n}{10(n-1)}}_{1/10} \Rightarrow \lim\limits_{n} n I_n = \frac{1}{10}$